# Classical Test Theory (CTT) for Assessing Reliability

- Topics:
  - Review of concepts and summary statistics
  - Characterizing differences between indicators
  - CTT-based assessments of reliability
    - Why alpha doesn't really matter
    - Why standard errors of measurement should matter

# Review: What are we trying to do?

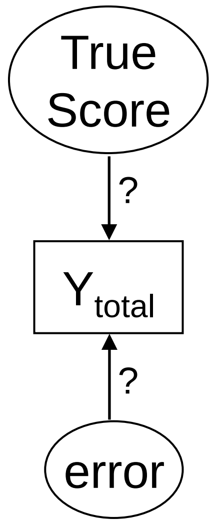
- Measure a **latent trait**: unobservable ability, characteristic, attitude, or other type of individually-varying construct
  - "**Latent**" = Not directly observable
  - "**Trait**" = true score, factor score, or theta as predictor(s) in measurement models; *aka*, latent construct, variable, or factor
  - The LTMMs we will cover are for **continuous latent traits**
- How to measure a latent trait? Collect **observed responses from indicators** chosen to measure the latent trait
  - "**Indicator**" = item, trial, or other response-specific outcome
  - Indicators can be any kind of variable (categorical or quantitative)
- How do we know we've done good job measuring the trait? **Collect evidence using the indicator responses**…
  - Two distinct ways such evidence gets used to represent a trait:
    - Build a **composite** (sum or average across indicator responses) → CTT
    - Use **all indicator responses** as outcomes of latent trait predictor instead: this is what happens in latent trait measurement models (LTMMs)

# Big Picture of Instrument Development

- Primary concerns about the use of an instrument to measure one or more latent traits have a **hierarchical structure**:
  - **Validity**: Extent to which an instrument measures what it is supposed to
    - Validity is always a matter of degree and depends critically on how it is used
    - Almost always demonstrated by **external evidence**: relationships to measures of other constructs in expected directions (e.g., discriminant and convergent validity)
  - An essential **precursor** to validity is **reliability**: Extent to which an instrument measures a latent trait with **sufficient consistency** (i.e., extent to which the same result would be obtained repeatedly)
    - "Validity is measuring the right thing; reliability is measuring the thing right"
    - Reliability indices will be provided differently across CTT and LTMMs (stay tuned)
  - An important **precursor** to reliability is **dimensionality**: Accuracy of the mapping of the observed indicators to the latent traits they measure
    - Reliability is per trait! Most reliability indices assume **unidimensional traits**
    - **What follows in this lecture presupposes that dimensionality is KNOWN!**

# Classical Test Theory (CTT)

- The **TOTAL** is the unit of analysis: $\boldsymbol{Y_{total} = True + Error}$
  - **True score $\boldsymbol{T}$:**
    - Best estimate of latent trait is **mean over infinite replications**
  - **Error $\boldsymbol{e}$:**
    - Expected value (mean) of 0; theoretically uncorrelated with $T$
    - Errors are supposed to wash out over repeated observations
  - **So the expected value of $\boldsymbol{T}$ is $\boldsymbol{Y_{total}}$**
    - This translates into $\boldsymbol{Y_{total}}$ = $\boldsymbol{T}$ true-score in practice
    - $\boldsymbol{Y_{total}}$ is referred to as a **total-score**, test-score, or scale-score




- Provides a framework with which to quantify ***reliability***
  - What proportion of **total-score** variance is due to **true-score** variance?
  - Understanding parts of CTT logic for quantifying reliability relies on traditional univariate and bivariate **summary statistics** for indicators…

# Means, Variances, Covariances, and Correlations

Using population notation: $N$ = # subjects, $s$ = subject, $i$ = item for $y_{is}$

**(Arithmetic) Mean ($\mu$):**
Central tendency of $y_{is}$

$$\mu_i = \frac{\sum_{s=1}^{N} y_{is}}{N}$$

**Variance ($Var$):**
Dispersion of $y_{is}$ in squared units

$$Var(y_i) = \sigma_{y_i}^2 = \frac{\sum_{s=1}^{N}(y_{is} - \bar{y}_i)^2}{N}$$

**Covariance ($Cov$):**
How outcomes (e.g., $y_{1s}$ and $y_{2s}$) go together in original metrics (unstandardized)

$$Cov(y_1, y_2) = \sigma_{y_1,y_2} = \frac{\sum_{s=1}^{N}[(y_{1s}-\bar{y}_1)(y_{2s}-\bar{y}_2)]}{N}$$

**Pearson Correlation ($r$):**
Covariance that has been standardized: $-1$ to 1

$$r(y_1, y_2) = \frac{Cov(y_1, y_2)}{\sqrt{Var(y_1)}\sqrt{Var(y_2)}}$$

# What about Categorical Indicators?

- Computing means, variances, covariances, and correlations is standard and intuitive for **quantitative indicators**
  - When the numbers are actually numbers (interval measurement)
  - e.g., magnitude estimation slider bars, response times
- But observed indicators are **more often categorical**:
  - Binary (i.e., dichotomous) → 2 options
  - Ordinal (i.e., "Likert scale") → 3+ ordered options
  - Nominal (i.e., multinomial) → 3+ unordered options
- For **nominal indicators**, means and variances make no sense…
  - Frequency of each category is needed instead (stay tuned)
  - But what about summarizing binary or ordinal indicators?

# Binary and Ordinal Indicators

- For **binary indicators ($y_{is}$ coded 0 or 1)**, variance is not a separate property (as it is in quantitative indicators)
  - If $\boldsymbol{p_i}$ = proportion of 1 values, and $\boldsymbol{q_i}$ = proportion of 0 values:
  - Mean $\boldsymbol{\mu_i} = \boldsymbol{p_i}$, $Var(\boldsymbol{y_i}) = \boldsymbol{p_i} * \boldsymbol{q_i}$ (same result even if computed as usual)

**Mean and Variance of a Binary Variable**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Mean ($\boldsymbol{p_i}$) | .0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1.0 |
| Variance | .0 | .09 | .16 | .21 | .24 | .25 | .24 | .21 | .16 | .09 | .0 |

- For **ordinal indicators**, you may see means and variances calculated as usual, but they should give you pause…
  - e.g., 1=Strongly Disagree, 2=Disagree, 3=Neutral, 4=Agree, 5=Strongly Agree…. could equally be 1, 20, 300, 4000, 50000
  - Maximum variance is limited by $k$ = # of response options used

$$Var_{max}(y_i) = \frac{(k-1)^2}{2}$$

# Differences Between Indicators

- All indicators can be characterized by **two properties** with respect to how they map onto the latent trait that they measure: **item difficulty** and **item discrimination**
  - Item = indicator, but the term "item" is always used in this context
  - Properties will be indexed differently across CTT and LTMMs
- **Item difficulty** is the indicator's **location** on the metric of the latent trait; also known as item "**severity**" for non-ability traits
  - i.e., an item of difficulty level X measures people at trait level X well
  - So, to measure people with a range of trait levels accurately, you need to include indicators that have a corresponding range of item difficulty
- **Item discrimination** is how **strongly the indicator relates** to the trait ("discrimination" is used for ability or non-ability traits)
  - Is the degree to which the **indicator differentiates among persons** in their latent traits (should be positive, and stronger is always better)

# Difficulty and Discrimination in CTT

- Under the belief that the best estimate of the **latent trait is the total-score** across indicators (i.e., sum or average) in CTT:

- **Item difficulty/severity** (location on the latent trait) is the **indicator's mean** across respondents
  - Only applicable to binary or quantitative items; also ordinal if you believe in the numbers (which is usually what people do in CTT)
  - Note that the difficulty terminology is conceptually backwards: An item with a higher mean is labeled as "higher difficulty" even though more people did well than not (so items with higher means are actually easier)
    - For this reason, I think it's ok to think of item means as indices of "easiness" instead
  - In LTMMs, difficulty/severity will become some kind of model intercept (which will break the problematic tie of respondents to indicators)

- Item difficulty/severity is often ignored in evaluating items in CTT, except when it causes problems with discrimination…

# Difficulty and Discrimination in CTT

- Under the belief that the best estimate of the **latent trait is the total-score** across indicators (i.e., sum or average) in CTT:

- **Item discrimination** (relationship to the latent trait) is the **indicator's Pearson correlation** with the total-score
  - Called "**item–total**" correlation; often replaced with "**item–remainder**" correlation (i.e., total without that item) so the correlation isn't inflated
  - Only applicable to binary or quantitative items; also to ordinal if you believe in the numbers (which is usually what people do in CTT)
  - In LTMMs, discrimination will become some kind of model slope

- Items of **extreme difficulty/severity** have a restricted range, which may result in **smaller item–total correlations**
  - Following common advice to remove extreme items will reduce your ability to measure respondents of corresponding extreme trait levels!

# Reliability of CTT Total-Scores

- Before and after screening/selecting items (i.e., an iterative process), a **total-score** is created: a sum or mean across indicator responses
  - The **total-score** is now the unit of analysis: $\boldsymbol{Y_{total} = True + Error}$
  - Even though the total-score doesn't know what kind of indicators were used to create it, the **total-score is always treated as a quantitative variable** (i.e., "ordinal-treated-as-interval")

- Then need to quantify **reliability**: the **consistency** with which $Y_{total}$ measures $True$ for a given respondent (i.e., subject)
  - Best index of $T$ for each subject is supposed to be the mean $Y_{total}$ over infinite replications… but that's not the kind of data usually collected!
  - Instead of *multiple replications* of total-score for a *single respondent*, more often collected are *single total-scores* for *multiple respondents*!
  - So reliability is instead defined using **between-subject sources** of respondent variance: $Reliability = Var(True)\ /\ Var(Y_{total})$
    - But to quantify reliability, we need more than one $Y_{total}$ per subject…

# How Only **Two Total-Scores** Can Yield a Reliability Coefficient in CTT

- $y_{1s} = T_s + e_{1s}$
- $y_{2s} = T_s + e_{2s}$

**CTT assumptions to calculate reliability:**

- Errors $e_{1s}$ and $e_{2s}$ have equal variance
- Total-scores $y_{1s}$ and $y_{2s}$ have equal variance
- Same subject-specific true score ($T_s$) at both times
- $e_{1s}$ and $e_{2s}$ are uncorrelated with each other and $T_s$

- Pearson Correlation between total-scores:
  - $r(y_1, y_2) = \frac{\sigma_{y_1,y_2}}{\sigma_{y_1}\sigma_{y_2}} = \frac{\sigma_{T+e_1,T+e_2}}{\sigma_{y_1}\sigma_{y_2}} = \frac{\sigma_{T,T} + \sigma_{T,e_1} + \sigma_{T,e_2} + \sigma_{e_1,e_2}}{\sigma_{y_1}\sigma_{y_2}} = \frac{\sigma_T^2}{\sigma_y^2}$

- In other words: $r(y_1, y_2) = Reliability = \ Var(True)\ /\ Var(Y_{total})$
  - So the Pearson correlation of two total-scores indexes how much of the observed total-score variance is due to "true" between-subject differences (if we believe all these untested assumptions)

# 3 Ways of Quantifying Reliability

- After measuring variance across subjects* two ways:
  1. Consistency of same test over time
     - Test-retest reliability
  2. Consistency over alternative test forms
     - Alternative forms reliability
     - Split-half reliability
  3. Consistency across items within a test
     - Internal consistency (alpha or KR-20)

** FYI: Some would say we have violated "ergodicity" by quantifying reliability in this between-subjects way:

- What factors cause differences between respondents is not the same as what factors causes differences within a respondent over occasions…

# 1. Test-Retest Reliability… What could go wrong?

- In a word, **CHANGE**: Test-retest reliability assumes that any difference in true-score is due to measurement error
  - Error = a characteristic of the test
  - It could be due to a characteristic of the person, too
- In a word, **MEMORY**: Assumes that testing procedure has no impact on a given subject's true-score, although:
  - Reactivity can lead to *higher* scores: learning, familiarity, memory…
  - Reactivity can lead to *lower* scores: fatigue, boredom…
- In a word (or two), **TEMPORAL INTERVAL**
  - Which test-retest correlation is the "right" one?
  - Should vary as a function of time (longer intervals → smaller correlation)
  - Long enough to limit memory, but short enough to avoid real change… how long is that, exactly????

# 2. Alternative Forms or Split-Half Reliability

- **Two forms of same test** administered "close" in time
  - Different indicators on each, but still measuring same construct
  - Forms need to be "**parallel**" – this means no systematic differences between in the summary statistics of the total-scores
    - Responses should differ ONLY because of random fluctuation (error)

- OR just take one test and **split it in half**! → Ta-da, two forms!
  - e.g., odd indicators = $y_{1s}$, even indicators = $y_{2s}$
  - BUT reliability is now based on half as many indicators!
  - What if we could **extrapolate** what reliability would be with twice as many indicators… Can do so using a reduced form of the "Spearman Brown Prophecy Formula" (assuming parallel indicators; stay tuned)
    - $Reliability_{new} = 2 * Reliability_{old} \;/\; (1 \;+ Reliability_{old})$
    - e.g., $Reliability_{old} = .75?\quad Reliability_{new} = 2 * .75 \;/\; 1.75 = .86$

# More about Two Total-Score Reliability… What could go wrong?

## Alternative Forms Reliability:

- In a word, **PARALLEL**:
  - Have to believe forms are sufficiently parallel: both total-scores have same mean, same variance, same true-scores and true-score variance, same error variance…
  - AND by extrapolation, all indicators within each test and across tests have equivalent psychometric properties and same correlations among them
  - Otherwise, indicator differences could create total-score differences
  - Still susceptible to problems caused by reactivity (change or retest effects)

## Split-Half Reliability:

- In a word (or two), **WHICH HALF**: There are many possible splits that would yield different reliability estimates… (e.g.,125 splits for 10 indicators)

# 3. Internal Consistency Reliability

- For quantitative indicators, this is usually **Cronbach's Alpha**…
  - Or "Guttman-Cronbach alpha" (Guttman 1945 < Cronbach 1951)
  - Equivalent form of alpha for binary items is named "KR 20"

- Alpha has been described in multiple ways:
  - Is the mean of all possible split-half correlations
  - As an index of "internal consistency"
    - Although Rod McDonald disliked this term… everyone else uses it

- Alpha is a lower-bound estimate of reliability under assumptions that all indicator responses:
  - Are **unidimensional** → MUST measure a single latent trait
  - Are **tau-equivalent → "true-score equivalent"**→ (sufficiently) equal item discrimination → equally related to the true score
  - Have **uncorrelated errors** (otherwise → multidimensional)

# Where Cronbach's Alpha comes from…

- The **sum of the $I$ indicator variances** (e.g., $I = 3$ here):
  - $\sum_{i=1}^{I} Var(y_i) = Var(y_1) + Var(y_2) + Var(y_3)$ → only the variances
  - Will become a baseline for expected amount of total-score variation

- ***Variance of the $I$ indicators' total-score*** is given by the sum the indicators' variances **PLUS their covariances**:
  - $Var(Y_{total}) = Var(y_1) + Var(y_2) + Var(y_3)$
    $+ \mathbf{2}Cov(y_1, y_2) + \mathbf{2}Cov(y_1, y_3) + \mathbf{2}Cov(y_2, y_3)$
  - Where does the **2** come from?
    - Covariance matrix is symmetric
    - Sum the whole thing to get to the *variance of the sum* of the indicators
  - So should be greater than sum of indicator variances above if they have something in common → covariance

| | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $y_1$ | $\sigma_{y_1}^2$ | $\sigma_{y_1,y_2}$ | $\sigma_{y_1,y_3}$ |
| $y_2$ | $\sigma_{y_1,y_2}$ | $\sigma_{y_2}^2$ | $\sigma_{y_2,y_3}$ |
| $y_3$ | $\sigma_{y_1,y_3}$ | $\sigma_{y_2,y_3}$ | $\sigma_{y_3}^2$ |

# Cronbach's Alpha: It's not what you think.

- $\boldsymbol{alpha}\ (\alpha) = \frac{I}{I-1} * \frac{Var(Y_{total}) - \sum_{i=1}^{I} Var(y_i)}{Var(Y_{total})}$

$I$ = # indicators

  - Numerator reduces to the indicator covariances → if the indicators are related, the variance of the indicators' total-score, $Var(y_{total})$, should be bigger than the sum of the indicator variances, $\sum_{i=1}^{I} Var(y_i)$
- Easier way: $\boldsymbol{alpha}(\alpha) = \frac{I\bar{r}}{1 + [\bar{r}(I-1)]}$

$\bar{r}$ = average inter-indicator Pearson correlation

  - Two ways to make alpha bigger: (1) Get more indicators, (2) increase the average inter-indicator correlation (but its's hard to do both at once)

- **Alpha** reliability assumes that all indicators are **unidimensional**
  - Formula does not take into account the spread of the inter-indicator correlations → **so alpha does NOT assess indicator dimensionality!**
- **Alpha** reliability assumes indicators have **equal discrimination** (tau-equivalent; equal relation to latent trait) with **uncorrelated errors**
  - Indicator properties are not included in the formula → exchangeable

# Alpha: What could go wrong?

- Alpha does not index **unidimensionality** → it does NOT index the extent to which the indicators measure the same construct

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**TABLE 18.2.** Interitem Correlation Matrices for Two Hypothetical Tests with the Same Coefficient Alpha Reliability of .81

Test A with 10 items

| Variable | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | – | | | | | | | | | |
| 2 | .3 | – | | | | | | | | |
| 3 | .3 | .3 | – | | | | | | | |
| 4 | .3 | .3 | .3 | – | | | | | | |
| 5 | .3 | .3 | .3 | .3 | – | | | | | |
| 6 | .3 | .3 | .3 | .3 | .3 | – | | | | |
| 7 | .3 | .3 | .3 | .3 | .3 | .3 | – | | | |
| 8 | .3 | .3 | .3 | .3 | .3 | .3 | .3 | – | | |
| 9 | .3 | .3 | .3 | .3 | .3 | .3 | .3 | .3 | – | |
| 10 | .3 | .3 | .3 | .3 | .3 | .3 | .3 | .3 | .3 | – |

Test B with 6 items

| Variable | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | – | | | | | |
| 2 | .6 | – | | | | |
| 3 | .6 | .6 | – | | | |
| 4 | .3 | .3 | .3 | – | | |
| 5 | .3 | .3 | .3 | .6 | – | |
| 6 | .3 | .3 | .3 | .6 | .6 | – |

- The *variability* across the inter-indicator correlations matters, too!
- We will use LTMMs predicting indicator responses to examine dimensionality

Example from: John, O. P., & Benet-Martinez, V. (2014). Measurement: Reliability, construct validation, and scale construction. In H. T. Reis & C. M. Judd (Eds.), *Handbook of research methods in social and personality psychology* (pp. 3473-503, 2nd ed.). New York, NY: Cambridge University Press.

# How to Get Alpha UP: More Items!

**Given indicator $\bar{r}$,**

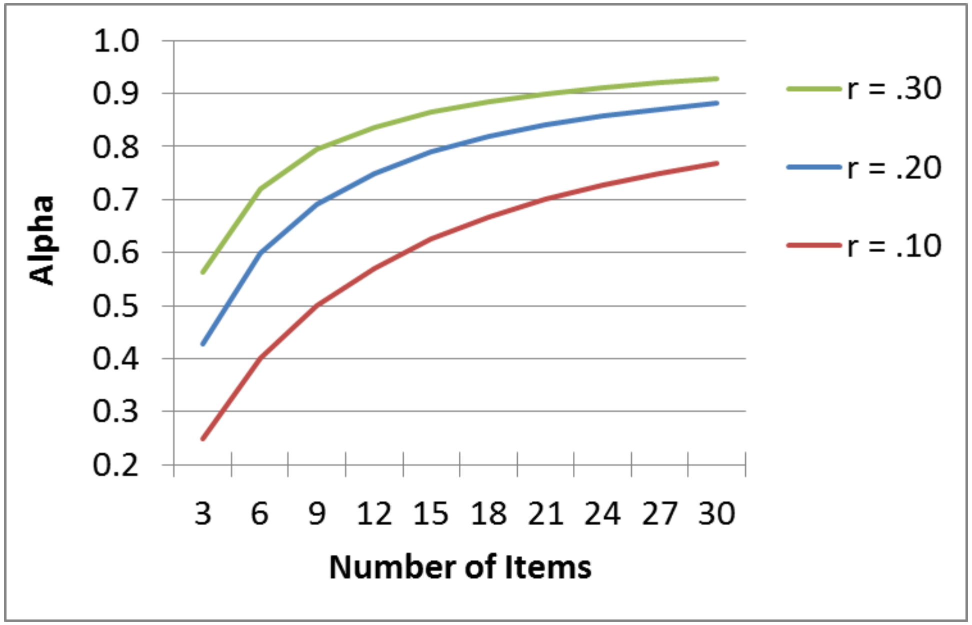
$$\boldsymbol{alpha} = \frac{I\bar{r}}{1 + [\bar{r}(I-1)]}$$

**Given alpha ($\alpha$),**

$$\bar{\boldsymbol{r}} = \frac{\alpha}{I - (\alpha I) + \alpha}$$

| Number of Indicators $I$ | Average Indicator $\bar{r}$ | | | |
|---|---|---|---|---|
| | **.2** | **.4** | **.6** | **.8** |
| **2** | .333 | .572 | .750 | .889 |
| **4** | .500 | .727 | .857 | .941 |
| **6** | .600 | .800 | .900 | .960 |
| **8** | .666 | .842 | .924 | .970 |
| **10** | .714 | .879 | .938 | .976 |

Btw: For the 2020 GRE psychology subject test, (KR-20) **alpha = .95**… for about 205 items, this means $\bar{\boldsymbol{r}} = \mathbf{.084!}$



# Kuder Richardson (KR) 20: Alpha for Binary Items (Indicators)

- From 'Equation 20' in 1937 paper:

$$\text{KR20} = \frac{k}{k-1}\left(\frac{\text{variance of total Y} - \text{sum of } pq \text{ over items}}{\text{variance of total Y}}\right)$$

$k$ = # items ($I$ before)
$p$ = proportion of 1s
$q$ = proportion of 0s

- Numerator again reduces to covariance among indicators…
  - **Sum of the indicator variances** (sum over indicators of $pq$) is just the variances
  - **Variance of the indicators' total-score** has their covariances in it, too
  - Numerator reduces to the indicator covariances → if the indicators are related, the variance of the sum of the indicators $Var(y_{total})$ should be bigger than the sum of the indicator variances $\sum_{i=1}^{I} Var(y_i)$
  - So KR20 is the same thing as alpha (it's just a computational shortcut)
  - Btw, this is how reliability is computed for the GRE subtests …

Kuder, G. F., & Richardson, M.W. (1937). The theory of the estimation of test reliability. *Psychometrika, 2*(3), 151–160.

# Limited Reliability of Binary Indicators

- The possible **Pearson's $r$ for binary variables will be limited** when they are not evenly split into 0/1 because their variance depends on their mean
  - Remember: Mean = $p_i$ , Variance = $p_i(1 - p_i) = p_i q_i$

- If two indicators ($x$ and $y$) differ in $p_i$, such that $p_y > p_x$
  - Maximum covariance: $Cov(x, y) = p_x(1 - p_y)$
  - This problem is known as **"range restriction"**
  - **Here this means the maximum Pearson's $r$ will be smaller than $\pm 1$ it should be:**

$$r_{x,y} = \sqrt{\frac{p_x(1-p_y)}{p_y(1-p_x)}}$$

  - Some examples using this formula to predict maximum Pearson $r$ values →
  - **So if indicator $\bar{r}$ is limited, so is reliability as measured by alpha (or KR-20)…**

| px | py | | max r |
|---|---|---|---|
| 0.1 | 0.2 | | 0.67 |
| 0.1 | 0.5 | | 0.33 |
| 0.1 | 0.8 | | 0.17 |
| 0.5 | 0.6 | | 0.82 |
| 0.5 | 0.7 | | 0.65 |
| 0.5 | 0.9 | | 0.33 |
| 0.6 | 0.7 | | 0.80 |
| 0.6 | 0.8 | | 0.61 |
| 0.6 | 0.9 | | 0.41 |
| 0.7 | 0.8 | | 0.76 |
| 0.7 | 0.9 | | 0.51 |
| 0.8 | 0.9 | | 0.67 |

# Correlations for Binary or Ordinal Indicators

- **Pearson correlation**: between two quantitative variables, working with the observed distributions as they actually are
- **Phi correlation**: between two binary variables, still working with the observed distributions (= Pearson with computational shortcut)
- **Point-biserial correlation**: between one binary and one quantitative variable, still working with the observed distributions (and still = Pearson)

*Line of Suspended Disbelief to Reduce Impact of Range Restriction*

- **Tetrachoric correlation**: between "underlying continuous" distributions of two actually binary variables (not = Pearson); aka, between probits
- **Biserial correlation**: between "underlying continuous" (but really binary) and observed quantitative variables (not = Pearson); aka, between probits
- **Polychoric correlation**: between "underlying continuous" distributions of two ordinal variables (not = Pearson); aka, between probits

- We will make use of **tetrachoric and polychoric correlations** in LTMMs predicting binary and ordinal indicator responses (limited-info estimation)

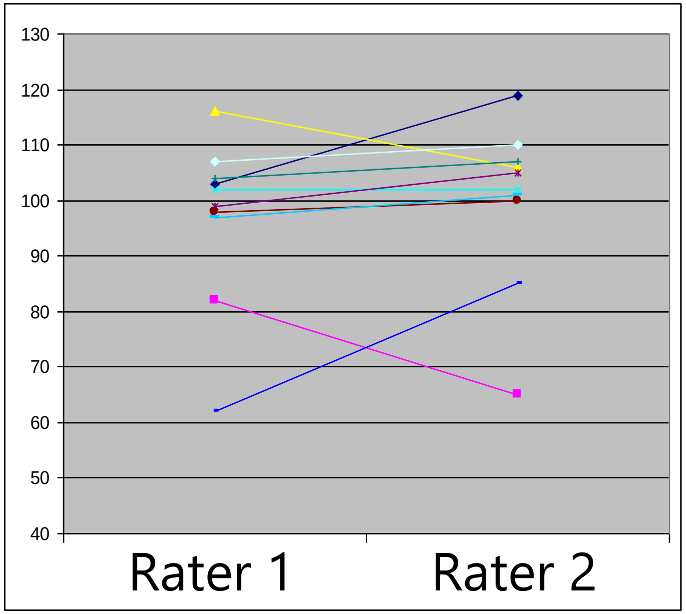
# More Correlations: Pearson vs. Intraclass

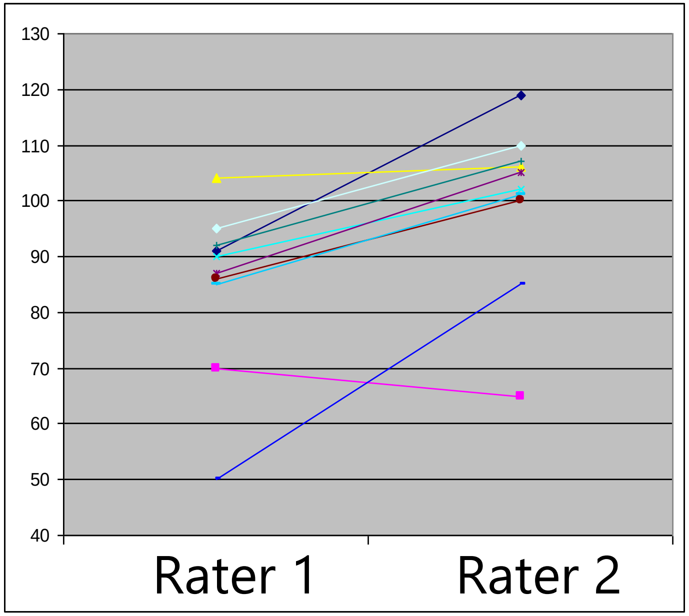
- **Pearson's $r$ is problematic for assessing reliability across raters**, because it ignores relevant differences in mean and variance across raters by standardizing each variable separately
  - e.g., **multiple raters** $(y_{1s}, y_{2s})$ each provide scores for the same set of targets

- Solution: use an "**Intraclass Correlation**" (ICC) instead, which standardizes across all raters using a **common mean and variance**
  - For example, for two raters: $\text{ICC}(y_1, y_2) = \frac{\sum_{s=1}^{N}[(y_{1s}-\bar{y})(y_{2s}-\bar{y})]}{(N-1)*s^2}$
    where $\bar{y} = \frac{\sum_{s=1}^{N}[(y_{1s}+y_{2s})]}{2N}$ and $s_y^2 = \frac{\sum_{s=1}^{N}(y_{1s}-\bar{y})^2 + \sum_{s=1}^{N}(y_{2s}-\bar{y})^2}{2N-1}$
  - ICC is also a ratio of variances: $ICC = \frac{s^2_{Between-Targets}}{s^2_{Between-Targets}+s^2_{Between-Raters}+s^2_{within-both}}$

- **ICCs can readily be extended** to more than two raters, as well as to quantify the effect of multiple distinct sources of sampling variance
  - e.g., multiple raters of multiple targets across days—how much variance is due to each?
  - Btw, this is the basis of "Generalizability Theory" (or G-Theory)—different variance components can be used to compute different reliability types (relative or absolute)

# Intraclass Correlation Example




| | Rater 1 | Rater 2 |
|---|---|---|
| $M$: | 97 | 100 |
| $SD$: | 15 | 15 |

$Pearson\ r = .670$

$Intraclass\ r = .679$

| | Rater 1 | Rater 2 |
|---|---|---|
| $M$: | 85 | 100 |
| $SD$: | 15 | 15 |

$Pearson\ r = .670$

$Intraclass\ r = .457$

$$ICC = \frac{s^2_{Between-Targets}}{s^2_{Between-Targets} + \boldsymbol{s^2_{Between-Raters}} + s^2_{within-both}}$$

# Reliability in a Perfect World, Part 1

- What would my reliability be if I just added more indicators?
- **Spearman-Brown Prophesy Formula**
  - $Reliability_{NEW} = \frac{ratio * reliability_{old}}{1+[(ratio-1)*reliability_{old}]}$

$$ratio = \frac{\#\ new\ indicators}{\#\ old\ indicators}$$

  - For example:
    - Old reliability = .40
    - Ratio = 5 times as many indicators (had 10, what if we had 50)
    - New reliability = .77

- To use this formula, you must assume **PARALLEL** indicators
  - All indicator discriminations equal, all indicator error variances equal, all covariances and correlations among indicators are equal, too
  - (Unlikely) assumption of parallel indicators is testable in LTMMs

# Assumptions about Indicators When Calculating Score Reliability in CTT

- Use of **alpha** as an index of reliability of total-scores requires an assumption of **tau-equivalent indicators**:
  - aka, "true-score equivalence" → equal item discrimination
  - Translates to **equal covariances** among indicators
    - But not necessarily equal correlations…(because different error variances)

- Use of **Spearman-Brown** Prophesy formula to predict new reliability requires an assumption of **parallel indicators**:
  - Tau-equivalent indicators PLUS equal error variances
  - This translates into equal correlations among indicators, too

- Btw, parallel indicators is also required to get a perfect correlation between latent trait estimates (of predictors as used in an LTMM) and total-scores as latent trait estimates in CTT
  - See McNeish & Wolf (2020) for constraints needed (on our syllabus)

# Reliability in a Perfect World, Part 2

- **Attenuation-corrected** correlations
  - What would our correlation between two latent traits be if our total-scores were "perfectly reliable"?
  - $r_{new} = r_{old}\sqrt{rel_x * rel_y}$ → all from same sample
  - For example:
    - Old correlation between $x$ and $y$: $r = .38$
    - $Reliability_x = .25$
    - $Reliability_y = .55$
    - New and "unattenuated" correlation: $r = 1.03$
  - Anyone see a problem here?
    - Btw—this logic forms the basis of SEM ☺

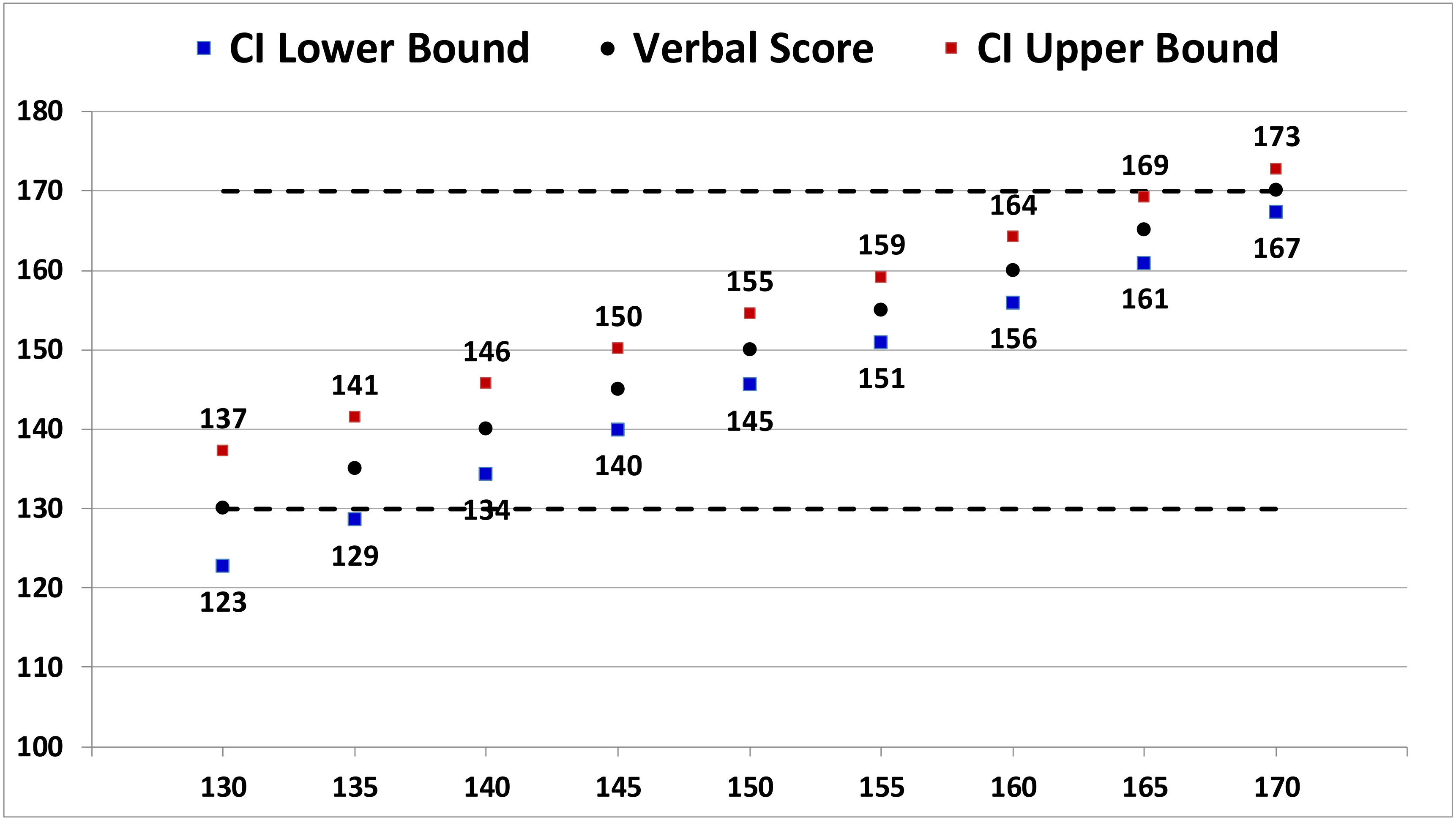
# Using Reliability Coefficients → SE

- **Reliability coefficients** ($Rel$) are sample-level statistics…
  - But reliability is a means to an end in interpreting a score for a **given individual**—we use reliability to get the **error variance**
  - $Var(True) = Var(Y_{total}) * Rel$; so $Var(Error) = Var(Y_{total}) - Var(True)$
  - $\boldsymbol{SD(error)}$ **is individual standard error of measurement,** $\boldsymbol{SE}$
  - **95% CI for individual total-score =** $\boldsymbol{Y_{total} \pm (1.96 * SE)}$
    - Gives precision of true score estimate in the metric of the original total-score

- e.g., if $Var(Y_{total}) = 100$ and $y_{total}$ for subject $s = 50$
  - $Rel = .91, Var(Error) = 9, SE = 3$ → $95\%\ CI \approx 44$ to $56$
    $Rel = .75, Var(Error) = 25, SE = 5$ → $95\%\ CI \approx 40$ to $60$
  - Note this assumes a symmetric distribution, and thus the limits of **CI can go out of bounds** of the scale for extreme scores
  - Note this also assumes the **SE for each person is the same!**
  - *Cue real-world example using the pre-pandemic GRE…*

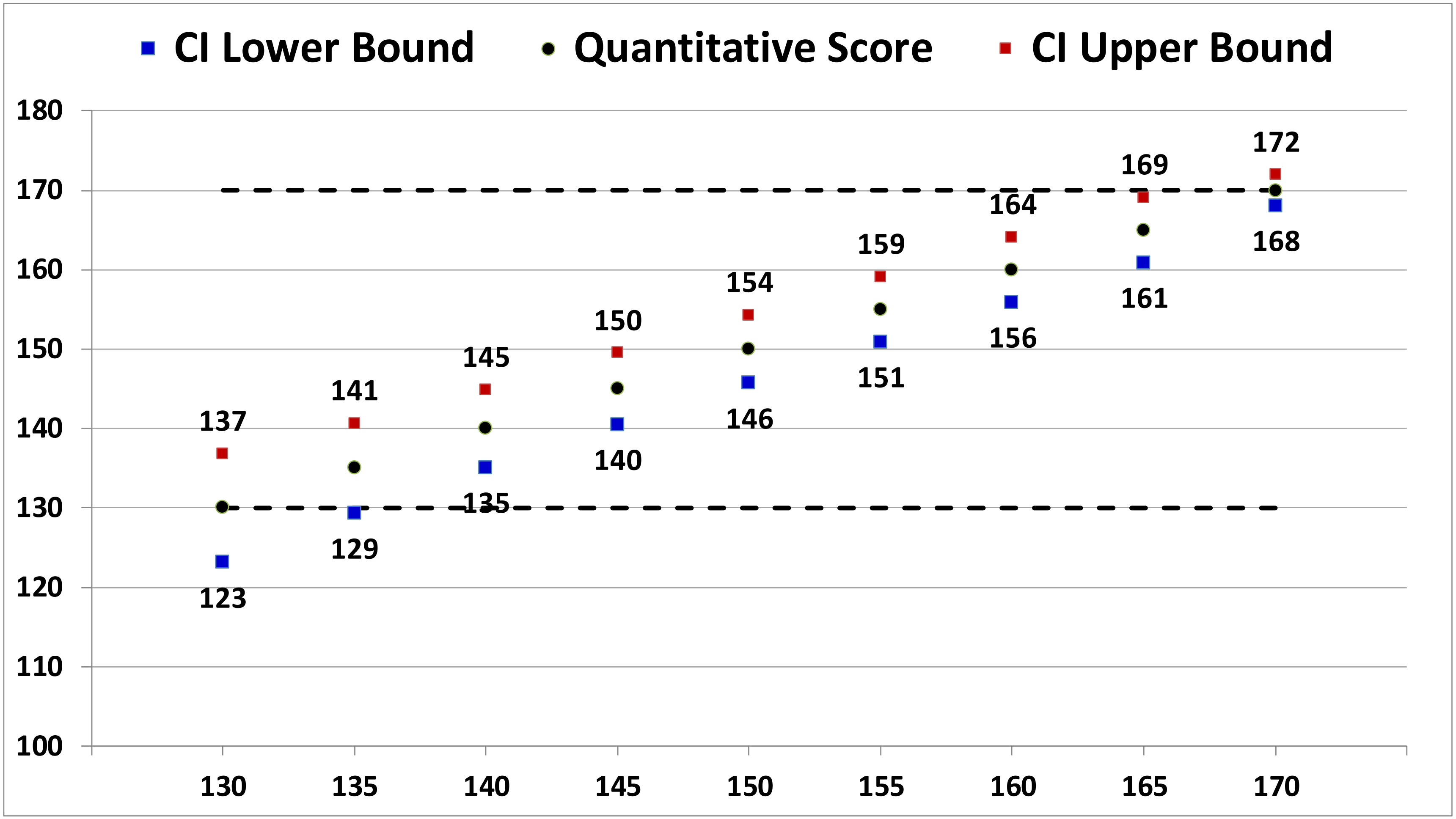
# 95% CIs for Individual Score: Verbal

## M=150.4, SD=8.5, range=130 to 170; SE=1.4 to 3.7



# 95% CIs for Individual Score: Quantitative

## M=153.4, SD=9.4, range=130 to 170; SE=1.0 to 3.5



# Intermediate Summary: CTT Reliability

- **CTT unit of analysis is the TOTAL:** $\boldsymbol{Y_{total} = True + Error}$
  - Total-score is best estimate of True Score (i.e., the Latent Trait)
  - I will call this an "ASU" measurement model (ASU = Add Stuff* Up)
    - ASU model assumes unidimensionality – the only thing that matters is the one $True$
  - Reliability of total-score cannot be quantified without assumptions that range from somewhat plausible to downright ridiculous (testable in item-level models)

- **Indicator responses are not included, which means:**
  - No way of explicitly testing dimensionality
  - Assumes all items are equally discriminating ("true-score-equivalent")
    - All items are equally related to the latent trait (also called "tau-equivalent")
  - To make a test better, you need more items
    - **What kind of items? More.**
  - Measurement error is assumed constant across the latent trait
    - **People low-medium-high in True Score are measured equally well**