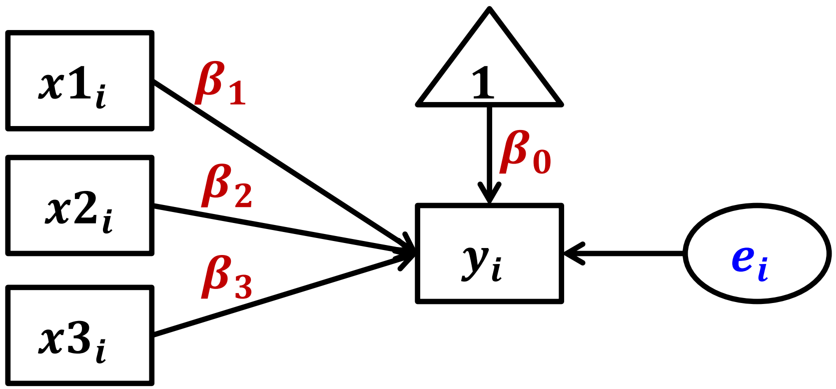
General Linear Models (GLMs) with Multiple Fixed Effects for a Single Predictor

- Topics:
  - Reviewing empty GLMs and single predictor GLMs
  - GLM special cases: 2+ fixed slopes to describe a predictor's effect
    - "Analysis of Variance" (ANOVA) for a one categorical predictor
      - e.g., income differences across 3 categories of employment class
    - Nonlinear effects of a single quantitative predictor
      - e.g., quadratic continuous effect of years of age on income
      - e.g., piecewise discontinuous effect of years of education on income
    - Testing linear effects of a single ordinal predictor
      - e.g., linear vs. nonlinear effect of 5-category happiness on income

Where we're headed in this unit…



This figure is a **path diagram**. This path diagram illustrates a general linear model with 3 x_i predictors of 1 y_i outcome. The "1" triangle is a constant used by the fixed intercept.

- **Synonyms for y_i outcome:** dependent variable, criterion, thing-to-be explained/predicted/accounted for
- **Synonyms for each x_i predictor:** regressor, independent variable (if manipulated), covariate (if quantitative or if it must be included to show incremental contributions above it)
  - This unit will cover the use of multiple predictors to describe the effect of a single conceptual predictor (next up is multiple conceptual predictors)
- Ways to describe the **goal of a model**:
  - "Examine effects of (the x_i predictors) on (the y_i outcome)"
  - "Regress (outcome y_i) on (the x_i predictors)"

Review: Empty Models and Single Predictor Models

- Predictive linear models create a **custom expected outcome** for each person through a linear combination of fixed effects that multiply predictor variables
  - $y_i = (\text{constant} * 1) + (\text{constant} * \text{Xpred1}_i) + (\text{constant} * \text{Xpred2}_i)\ldots$

- <u>Empty GLM:</u> **Actual** $\boldsymbol{y_i = \beta_0 + e_i}$, **Predicted** $\boldsymbol{\hat{y}_i = \beta_0}$
  - $\boldsymbol{\beta_0}$ = **intercept** = expected $\boldsymbol{y_i}$ = here is mean $\bar{y}$ (best naïve guess if no predictors)
  - $\boldsymbol{e_i}$ = **residual** = is always the deviation between the actual $\boldsymbol{y_i}$ and predicted $\boldsymbol{\hat{y}_i}$
    - Because $\boldsymbol{\hat{y}_i}$ = $\bar{y}$ for all, the $\boldsymbol{e_i}$ residual variance across persons ($\boldsymbol{\sigma_e^2}$) is **all the** $\boldsymbol{y_i}$ **variance**

- <u>Add a predictor:</u> **Actual** $\boldsymbol{y_i = \beta_0 + \beta_1(x_i) + e_i}$, **Predicted** $\boldsymbol{\hat{y}_i = \beta_0 + \beta_1(x_i)}$
  - $\boldsymbol{\beta_0}$ = **intercept** = expected $\boldsymbol{y_i}$ when $\boldsymbol{x_i = 0}$ (so always ensure $x_i = 0$ makes sense)
  - $\boldsymbol{\beta_1}$ = **slope of** $\boldsymbol{x_i}$ = difference in $\boldsymbol{y_i}$ per one-unit difference in $\boldsymbol{x_i}$
  - $\boldsymbol{e_i}$ = **residual** = is always the deviation between the actual $\boldsymbol{y_i}$ and predicted $\boldsymbol{\hat{y}_i}$
    - Now $\boldsymbol{\hat{y}_i}$ differs by $\boldsymbol{x_i}$, so $\boldsymbol{e_i}$ residual variance across persons ($\boldsymbol{\sigma_e^2}$) is **leftover** $\boldsymbol{y_i}$ **variance**

1 Fixed Effect for a Single Predictor

- $\boldsymbol{\beta_1}$ for the **slope of** $\boldsymbol{x_i}$ is scale-specific → is "unstandardized"
- Unstandardized results for $\boldsymbol{\beta_1}$ include:
  - **Estimate** = (Est) = most likely value for the sample's slope
  - **Standard Error** (SE) = index of inconsistency across samples = how far away on average a sample $\boldsymbol{x_i}$ slope is from the population $\boldsymbol{x_i}$ slope
    - With only a single slope in the model, the SE for its estimate depends on the model residual variance ($\boldsymbol{\sigma_e^2}$), variance of $\boldsymbol{x_i}$ (σ_x^2), and $DF_{denominator}$: sample size minus k, the number of $\boldsymbol{\beta}$ model fixed effects (N − k)
  - **Test-statistic** $\boldsymbol{t}$ = $(Est - H_0)/SE$ → "**Univariate Wald test**" gives p-value for slope's significance using t-distribution and $DF_{denominator} = N - k$

- Can also request a "standardized" slope to provide an $\boldsymbol{r}$ **effect size**:
  - For a GLM with a **single** quantitative or binary predictor, $\beta_{std} = Pearson\ r$

$$\beta_{std} = \boldsymbol{\beta_{unstd}} * \frac{SD_x}{SD_{\boldsymbol{y}}}$$

GLMs with Predictors: Binary vs. 3+ Categories

- To examine a **binary predictor** of a quantitative outcome, we only need **2 fixed effects** to tell us **3 things**: the outcome mean for Category=0, the outcome mean for Category=1, and the outcome mean difference
- **Actual** $y_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{Category_i}) + \boldsymbol{e_i}$, **Predicted** $\hat{y}_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{Category_i})$
  - Category 0 Mean: $\hat{y}_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\mathbf{0}) = \boldsymbol{\beta_0}$ ← fixed effect #1
  - Difference of Category 1 from Category 0: $(\boldsymbol{\beta_0} + \boldsymbol{\beta_1}) - (\boldsymbol{\beta_0}) = \boldsymbol{\beta_1}$ ← fixed effect #2
  - Category 1 Mean: $\hat{y}_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\mathbf{1}) = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}$ ← linear combination of fixed effects
  - To get the estimate and SE for any mean created from a linear combination of fixed effects, you need to ask for it via SAS ESTIMATE, STATA LINCOM, or R GLHT
  - Btw, this type of GLM is also called a "two-sample" or "independent groups" *t*-test

- To examine the effect of a **predictor** with 3+ categories, the GLM needs **as many fixed effects as the number of predictor variable categories =** $\boldsymbol{C}$
  - If $\boldsymbol{C} = \mathbf{3}$, then we need the $\boldsymbol{\beta_0}$ intercept and 2 predictor slopes: $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$
  - If $\boldsymbol{C} = \mathbf{4}$, then we need the $\boldsymbol{\beta_0}$ intercept and 3 predictor slopes: $\boldsymbol{\beta_1}$, $\boldsymbol{\beta_2}$, and $\boldsymbol{\beta_3}$
  - # pairwise mean differences = $\frac{C!}{2!(C-2)!}$ → e.g., given $\boldsymbol{C} = \mathbf{3}$, # diffs = $\frac{3*2*1}{(2*1)(1)} = 3$
  - This type of GLM goes by the name "**Analysis of Variance**" (**ANOVA**) in which the term "category" is usually replaced with "group" as a synonym

"Indicator Coding" for a 3-Category Predictor

- Comparing the means of a quantitative outcome across **3 categories** requires creating **2 new binary predictors** to be included **simultaneously** along with the intercept, for example, as coded so Low= Intercept (ref)

workclass variable ($N = 734$)	LvM: Low vs Mid?	LvU: Low vs Upp?
1. Low ($n = 436$)	0	0
2. Mid ($n = 278$)	**1**	0
3. Upp ($n = 20$)	0	**1**

Actual: $y_i = \beta_0 + \beta_1(LvM_i) + \beta_2(LvU_i) + e_i$
Predicted: $\hat{y}_i = \beta_0 + \beta_1(LvM_i) + \beta_2(LvU_i)$

- Model-implied means per category (group):
  - Low Mean: $\hat{y}_L = \beta_0 + \beta_1(0) + \beta_2(0) = \beta_0$ ← fixed effect #1
  - Mid Mean: $\hat{y}_M = \beta_0 + \beta_1(1) + \beta_2(0) = \beta_0 + \beta_1$ ← found as linear combination
  - Upp Mean: $\hat{y}_U = \beta_0 + \beta_1(0) + \beta_2(1) = \beta_0 + \beta_2$ ← found as linear combination
- Model-implied differences between each pair of categories (groups):
  - Diff of Low vs Mid: $(\beta_0 + \beta_1) - (\beta_0) = \beta_1$ ← fixed effect #2
  - Diff of Low vs. Upp: $(\beta_0 + \beta_2) - (\beta_0) = \beta_2$ ← fixed effect #3
  - Diff of Mid vs Upp: $(\beta_0 + \beta_2) - (\beta_0 + \beta_1) = \beta_2 - \beta_1$ ← found as linear combination

See p. 278 of: Darlington, R. B., & Hayes, A. F. (2016). *Regression analysis and linear models: Concepts, applications, and implementation*. Guilford.

GLM 3-Category Predictor: Results

Empty Model: $y_i = \beta_0 + e_i$

- Model parameters:
  - Intercept β_0: $Est = 17.30$ $SE = 0.51$
  - Residual Variance σ_e^2: $Est = 190.21$

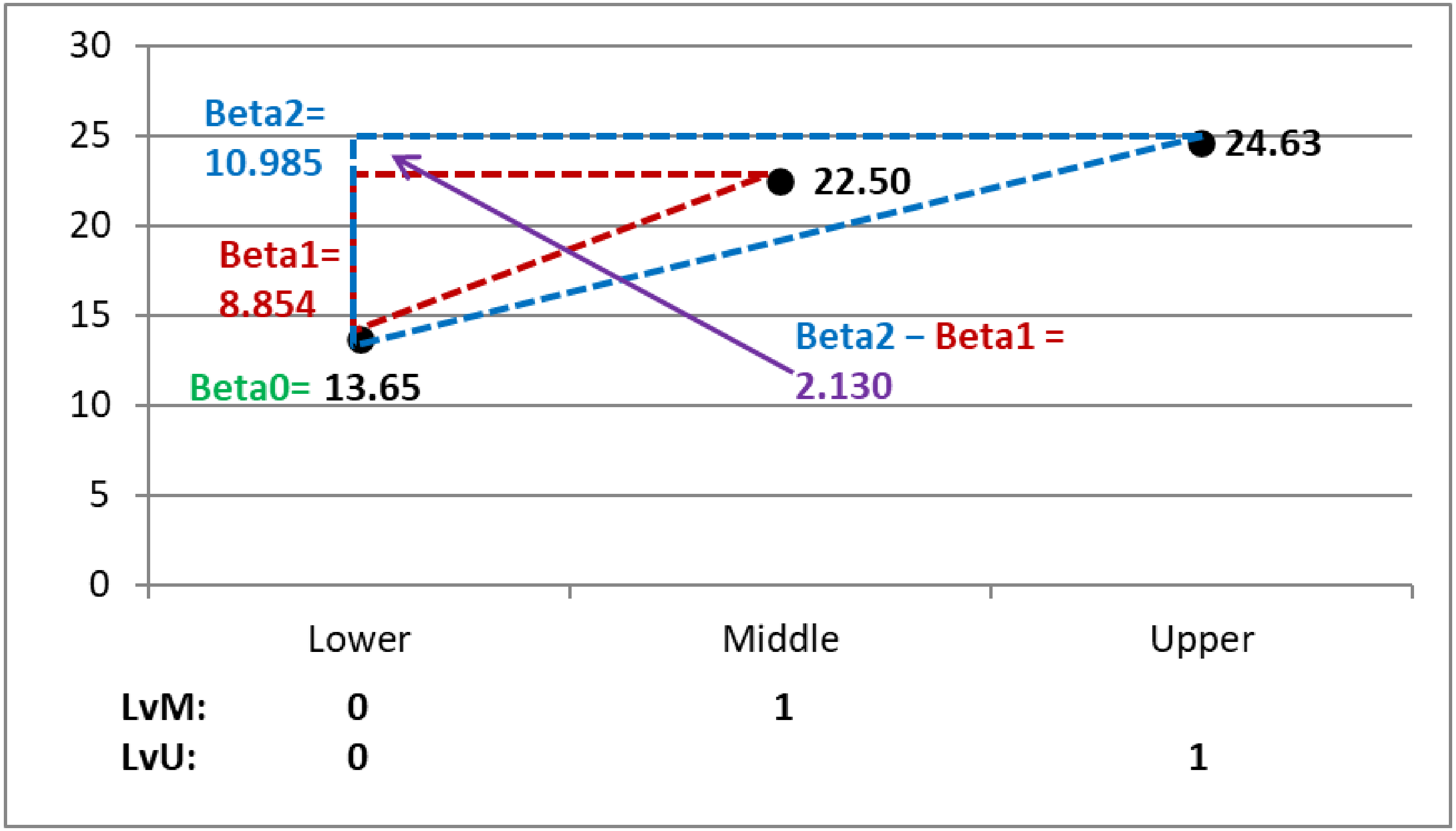
Group ($N = 734$)	LvM	LvU
1. Low ($n = 436$)	0	0
2. Mid ($n = 278$)	**1**	0
3. Upp ($n = 20$)	0	**1**

Predictor Model: $y_i = \beta_0 + \beta_1(LvM_i) + \beta_2(LvU_i) + e_i$

- Model parameters:
  - Intercept β_0: $Est = 13.65, SE = 0.63, p < .001$ → Mean for Low ($= \hat{y}_L$)
  - Slope β_1: $Est = 8.85, SE = 1.00, p < .001$ → Mean diff for Low vs Mid
  - Slope β_2: $Est = 10.98, SE = 2.99, p < .001$ → Mean diff for Low vs Upp
  - Residual Variance σ_e^2: $Est = 171.01$
- Linear combinations of model parameters:
  - Mid Mean: $\hat{y}_M = 13.65 + 8.85(1) + 10.98(0) = 22.50, SE = 0.78, p < .001$
  - Upp Mean: $\hat{y}_U = 13.65 + 8.85(0) + 10.98(1) = 24.63, SE = 2.92, p < .001$
  - Mean diff of Mid vs Upp $= \beta_2 - \beta_1 = 2.13, SE = 3.03, p = .482$

GLM 3-Category Predictor: Results

Fixed Effects				Predictors				Category	Pred
Beta0	Beta1	Beta2		Intercept	LvM	LvU		workclass	Y Hat
13.650	8.854	10.985		1	0	0		Lower	13.65
13.650	8.854	10.985		1	1	0		Middle	22.50
13.650	8.854	10.985		1	0	1		Upper	24.63



Example with a 4-Category Predictor

Comparing outcome means across **4 groups** requires creating **3 new binary predictors** to be included **simultaneously** along with the intercept—for example, using "**indicator dummy-coded**" **predictors** so Control= Reference

Treatment Group	d1: C vs T1?	d2: C vs T2?	d3: C vs T3?
1. Control	0	0	0
2. Treatment 1	**1**	0	0
3. Treatment 2	0	**1**	0
4. Treatment 3	0	0	**1**

- Model: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{d1_i}) + \boldsymbol{\beta_2}(\boldsymbol{d2_i}) + \boldsymbol{\beta_3}(\boldsymbol{d3_i}) + \boldsymbol{e_i}$
  - The model gives us **the predicted outcome mean for each category** as follows:

Control (Ref) Mean	Treatment 1 Mean	Treatment 2 Mean	Treatment 3 Mean
$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_0}+\boldsymbol{\beta_1}(\boldsymbol{d1_i})$	$\boldsymbol{\beta_0}+\boldsymbol{\beta_2}(\boldsymbol{d2_i})$	$\boldsymbol{\beta_0}+\boldsymbol{\beta_3}(\boldsymbol{d3_i})$

  - Model directly provides 3 mean differences (control vs. each treatment), and indirectly provides another 3 mean differences (differences between treatments) as **linear combinations of the fixed effects**… let's see how this works

See p. 278 of: Darlington, R. B., & Hayes, A. F. (2016). *Regression analysis and linear models: Concepts, applications, and implementation*. Guilford.

Example with a 4-Category Predictor

Control (Ref) Mean = 10	Treatment 1 Mean = 12	Treatment 2 Mean = 15	Treatment 3 Mean = 19
$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_0}+\boldsymbol{\beta_1}(\boldsymbol{d1_i})$	$\boldsymbol{\beta_0}+\boldsymbol{\beta_2}(\boldsymbol{d2_i})$	$\boldsymbol{\beta_0}+\boldsymbol{\beta_3}(\boldsymbol{d3_i})$

Model: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{d1_i}) + \boldsymbol{\beta_2}(\boldsymbol{d2_i}) + \boldsymbol{\beta_3}(\boldsymbol{d3_i}) + \boldsymbol{e_i}$

Given means above, here are the pairwise category differences:

	Alt Group		Ref Group	Difference
• C vs. T1 =	$(\beta_0+\boldsymbol{\beta_1})$	$-$	(β_0)	$= \boldsymbol{\beta_1} = 2$
• C vs. T2 =	$(\beta_0+\boldsymbol{\beta_2})$	$-$	(β_0)	$= \boldsymbol{\beta_2} = 5$
• C vs. T3 =	$(\beta_0+\boldsymbol{\beta_3})$	$-$	(β_0)	$= \boldsymbol{\beta_3} = 9$
• T1 vs. T2 =	$(\beta_0+\boldsymbol{\beta_2})$	$-$	$(\beta_0+\boldsymbol{\beta_1})$	$= \boldsymbol{\beta_2} - \boldsymbol{\beta_1} = 5 - 2 = 3$
• T1 vs. T3 =	$(\beta_0+\boldsymbol{\beta_3})$	$-$	$(\beta_0+\boldsymbol{\beta_1})$	$= \boldsymbol{\beta_3} - \boldsymbol{\beta_1} = 9 - 2 = 7$
• T2 vs. T3 =	$(\beta_0+\boldsymbol{\beta_3})$	$-$	$(\beta_0+\boldsymbol{\beta_2})$	$= \boldsymbol{\beta_3} - \boldsymbol{\beta_2} = 9 - 5 = 4$

Back to the 3-Category Predictor GLM

- The ANOVA-type question "Does group membership predict y_i?" translates to "Are there significant group mean differences in y_i"?

  - Can be answered **<u>specifically</u> via <u>pairwise</u> group differences** given directly by (or created from) the model fixed effects: For example: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{LvM_i}) + \boldsymbol{\beta_2}(\boldsymbol{LvU_i}) + \boldsymbol{e_i}$,
    - Is $\boldsymbol{\beta_1} \neq 0$? If so, then $\boldsymbol{\hat{y}_M} \neq \boldsymbol{\hat{y}_L}$ (given directly because of our coding)
    - Is $\boldsymbol{\beta_2} \neq 0$? If so, then $\boldsymbol{\hat{y}_U} \neq \boldsymbol{\hat{y}_L}$ (given directly because of our coding)
    - Is $(\boldsymbol{\beta_2} - \boldsymbol{\beta_1}) \neq 0$? If so, then $\boldsymbol{\hat{y}_U} \neq \boldsymbol{\hat{y}_M}$ (requested as linear combination)

  - A **more <u>general</u> answer** to "**Does group matter?**" requires testing if $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ differ from 0 **<u>jointly</u>**, in other words:
    - Is the **residual variance** from this model with two grouping predictors **significantly lower** than the total variance from the empty model?
    - Does the **predicted** $\boldsymbol{\hat{y}_i}$ provided by this model with two group predictors **correlate significantly with the actual $\boldsymbol{y_i}$?**

Prediction Gained vs. DF spent

- To provide a **more general answer** to "**Does group matter?**" we need to consider the impact of our prediction <u>relative to</u> how many fixed effects we needed to generate predicted $\hat{y}_i$ and how good they did (relative to what is left unknown)
  - This is an example of a "**multivariate Wald test**" (stay tuned for others)
  - "Relative" is quantified using two types of **Degrees of Freedom** = $\boldsymbol{DF}$ = total number of fixed effects possible → total DF = sample size N
    - "$DF_{numerator}$" = $k-1$ = number of fixed <u>slopes</u> in the model
    - "$DF_{denominator}$" = number of DF left over (not yet spent): $N-k$
  - In GLMs, the amount of information captured by the model's prediction and the amount of information left over are quantified using different sources of "**sums of squares**" ($\boldsymbol{SS}$)
    - Basic form of $\boldsymbol{SS}$ is the **numerator** in computing variance: $\frac{\sum_{i=1}^{N}(y_i-\bar{y})^2}{N-1}$
    - For example, "**outcome (or total) $\boldsymbol{SS}$**" = $\boldsymbol{SS_{total}} = \sum_{i=1}^{N}(y_i-\bar{y})^2$

Prediction Gained vs. DF spent

- How much information is provided by our **model prediction** is quantified by "**model sums of squares**": $\boldsymbol{SS_{model}} = \sum_{i=1}^{N}(\boldsymbol{\hat{y}_i} - \boldsymbol{\bar{y}})^2$
- To quantify the **relative** size of that predicted info, we need to adjust it for $DF_{numerator}$ = number of fixed <u>slopes</u> = $k - 1$
  - Then get "**Model Mean Square**" = $\boldsymbol{MS_{model}} = \frac{SS_{model}}{k-1}$ — -1 because intercept doesn't get counted
  - $\boldsymbol{MS_{model}}$ = "how much information has been captured per point spent"

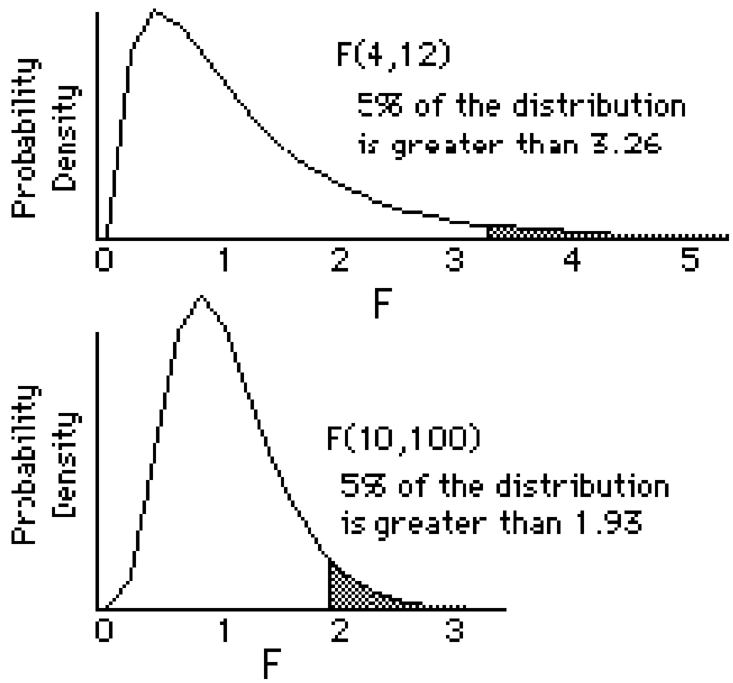
- How much information is **leftover** is quantified by "**residual** (or **error**) **sums of squares**": $\boldsymbol{SS_{residual}} = \sum_{i=1}^{N}(\boldsymbol{y_i} - \boldsymbol{\hat{y}_i})^2$
- To quantify the relative size of that leftover information, we need to adjust it for $DF_{denominator}$ = $N - k$
  - "**Residual** (or **Error**) **Mean Square**" = $\boldsymbol{MS_{residual}} = \frac{SS_{residual}}{N-k}$
  - $\boldsymbol{MS_{residual}}$ = "how much information left to explain per point remaining"

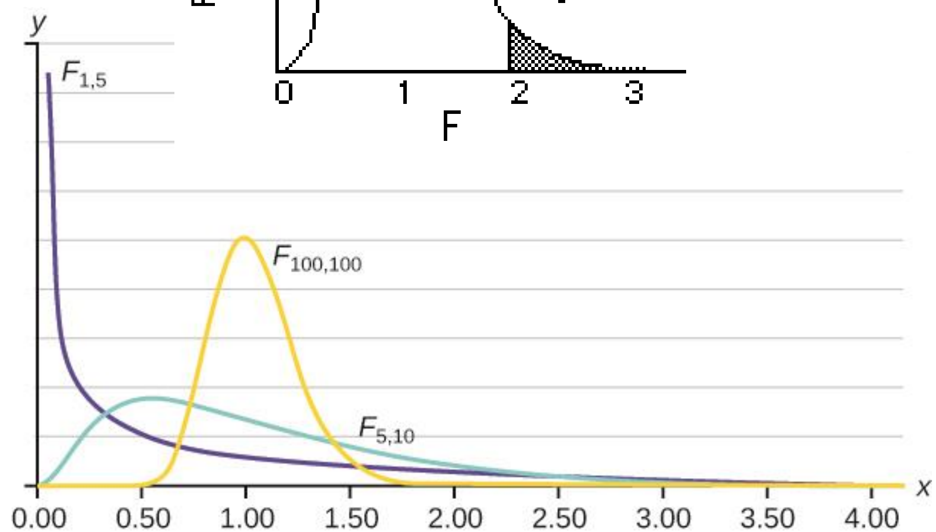
Prediction Gained vs. DF spent

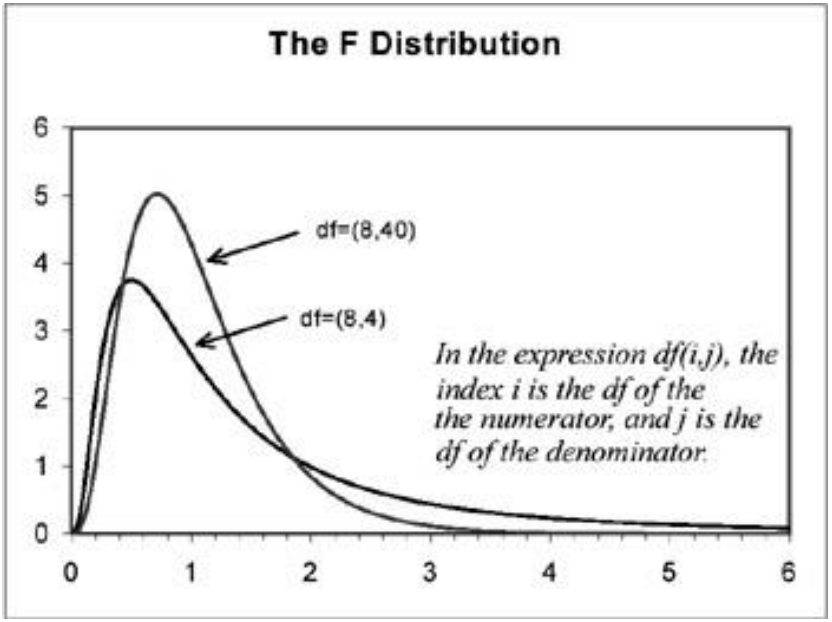
Source of Outcome Information	Sums of Squares (each summed from $i = 1$ to N)	Degrees of Freedom	Mean Square
Model (**known** because of predictor slopes)	SS_{model}: $(\hat{y}_i - \bar{y})^2$	DF_{num}: $k - 1$	MS_{model}: $\frac{SS_{model}}{k-1}$
Residual (leftover after predictors; still **unknown**)	$SS_{residual}$: $(y_i - \hat{y}_i)^2$	DF_{den}: $N - k$	$MS_{residual}$: $\frac{SS_{residual}}{N-k}$
"Corrected" Total (all **original** information in y_i)	SS_{total}: $(y_i - \bar{y})^2$	DF_{total}: $N - 1$ **(not shown)**	MS_{total}: $\frac{SS_{total}}{N-1}$ **(not shown)**

- This table now provides us with a way to answer the more general question of "Does group membership predict y_i?" → **Is our <u>model</u> significant**?
  - **Variance explained** by model fixed slopes: $R^2 = \frac{SS_{total} - SS_{residual}}{SS_{total}}$
  - R^2 = square of correlation between model-predicted $\hat{y}_i$ and actual y_i
  - F **test-statistic** for significance of $R^2 > 0$**?** is given two equivalent ways: $F(DF_{num}, DF_{den}) = \frac{MS_{model}}{MS_{residual}}$ or $F(DF_{num}, DF_{den}) = \frac{(N-k)R^2}{(k-1)(1-R^2)}$

Your New Friend, the F distribution







- The F test-statistic (F-value) is a ratio (in a squared metric) of “info explained over info unknown”, so ***F*-values must be positive**
- Its shape (and thus the critical value for the boundary of where “expected” starts) varies by DF_{num} (like χ^2) and by DF_{den} (like t, which is flatter for smaller $N - \mathrm{k}$)

Top left image borrowed from: https://www.statsdirect.com/help/distributions/f.htm
Top right image borrowed from: https://www.globalspec.com/reference/69569/203279/11-9-the-f-distribution
Bottom left image borrowed from: https://www.texasgateway.org/resource/133-facts-about-f-distribution

Summary: Steps in Significance Testing

- **Choose critical region: % alpha ("unexpected") and possible direction**
  - Both directions or just one?
  - Alpha (α) (1 −% confidence)?
  - Distribution for test-statistic will be dictated as follows:

Uses Denominator Degrees of Freedom?	Test 1 slope*	Test >1 slope*
No: implies infinite N	z	$\chi^2 (= z^2$ if 1)
Yes: adjusts based on N	t	$\boldsymbol{F} (= t^2$ if 1)

- If the **test-statistic exceeds** the distribution's critical value(s), then the obtained ***p*-value is less than the chosen alpha** level:
  - You "**reject the null hypothesis**"—it is sufficiently **unexpected** to get a test-statistic that extreme *if the null hypothesis is true;* result is "**significant**"

- If the **test-statistic does NOT exceed** the distribution's critical value(s), then the ***p*-value is greater than or equal to the chosen alpha** level:
  - You "**DO NOT reject the null hypothesis**"—it is sufficiently **expected** to get a test-statistic that extreme *if the null hypothesis is true;* result is "**not significant**"

* # Fixed slopes (or associations) = **numerator degrees of freedom** = $k - 1$

Significance of the Model Prediction

- With **only 1 predictor**, we don't need a separate F test-statistic of the model R^2 significance; for example: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{x_i}) + \boldsymbol{e_i}$
  - Significance of unstandardized $\boldsymbol{\beta_1}$ comes from $\boldsymbol{t} = (\boldsymbol{Est} - \boldsymbol{H_0})/\boldsymbol{SE}$
    - Significance of the model prediction $\boldsymbol{R^2}$ from $\boldsymbol{F} = \boldsymbol{t^2}$ already
    - So if $\boldsymbol{\beta_1}$ is significant via $|\boldsymbol{t_{\beta_1}}| > \boldsymbol{t_{critical}}$, then the $\boldsymbol{F}$ **test-statistic** for the model is significant, too → sufficiently unexpected if H_0 were true
  - Standardized $\boldsymbol{\beta_1}$ = Pearson's $\boldsymbol{r}$ between predicted $\boldsymbol{\hat{y}_i}$ and actual $\boldsymbol{y_i}$
    - So model $\boldsymbol{R^2} = \frac{SS_{total} - SS_{residual}}{SS_{total}}$ is the same as **(Pearson's $\boldsymbol{r}$)^2**

- With **2+ fixed slopes**, we DO need to examine model $\boldsymbol{F}$ test-statistic and $\boldsymbol{R^2}$, for example: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{LvsM_i}) + \boldsymbol{\beta_2}(\boldsymbol{LvsU_i}) + \boldsymbol{e_i}$
  - $\boldsymbol{F}$ test-statistic: Is the $\boldsymbol{\hat{y}_i}$ predicted from $\boldsymbol{\beta_1}$ AND $\boldsymbol{\beta_2}$ together significantly correlated with actual $\boldsymbol{y_i}$? The square of that correlation is the **model $\boldsymbol{R^2}$**
  - F test-statistic evaluates model R^2 *per DF spent to get it and DF leftover*

Significance of the Model: Example

- For example: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{LvsM_i}) + \boldsymbol{\beta_2}(\boldsymbol{LvsU_i}) + \boldsymbol{e_i}$
- Group-specific results: We already know that L<M, L<U, and L=M
- Significance test of the model: $R^2 = .103$
- **Report as $F(DF_{num}, DF_{den}) = Fvalue,\ \text{MSE} = MS_{res},\ p < pvalue$**

Source of Outcome Information	Sums of Squares (each summed from $i = 1$ to N)	Degrees of Freedom	Mean Square	F Value
Model (**known**)	$SS_{model}: (\hat{y}_i - \bar{y})^2 = 14{,}414.03$	$DF_{num}: k-1$ = 2 slopes (−1 for int)	$MS_{model}: \frac{SS_{model}}{k-1} = 7{,}207.01$	42.14
Residual (**"error"**)	$SS_{residual}: (y_i - \hat{y}_i)^2 = 125{,}009.25$	$DF_{den}: N-k$ = 731 leftover	$MS_{residual}: \frac{SS_{residual}}{N-k} = 171.01$	
Corrected Total (after $\bar{y}$)	$SS_{total}: (y_i - \bar{y})^2 = 139{,}423.23$	$N = 734 - 1$ = 733 total corrected for int		

Another version of R^2: "Adjusted R^2"

- Just like we may want to adjust Pearson's r for bias due to small sample size, some feel the need to **adjust the model R^2**
  - $R^2 = \frac{SS_{total} - SS_{residual}}{SS_{total}}$ → Must be positive if computed this way
  - $\boldsymbol{R^2_{adj}} = 1 - \frac{(1-R^2)(N-1)}{N-k-1} = 1 - \frac{MS_{residual}}{MS_{total}}$ → Change in residual variance relative to empty model
    - R^2_{adj} can be negative! (i.e., for a really-not-useful set of fixed slopes)

- Although adjusted R^2 is considered as the only "correct" version by a few, I have never once been asked to report it…
  - But just in case Reviewer 3 wants it some day, here you go…
  - For our example: $R^2_{adj} = 1 - \frac{(1-.103)(734-1)}{734-3} = .101$ ($R^2_{unadj} = .103$)
  - Btw, we need to use SAS PROC REG instead of SAS PROC GLM to get R^2_{adj} (both R^2 versions are given by STATA REGRESS and R LM)

Effect Size per Fixed Slope

- The **model R^2 value** (the square of the correlation between predicted $\hat{y}_i$ and actual y_i) provides a **general effect size**, but you may also want an **effect size for each fixed slope**
  - Why? To standardize the effect magnitude and/or to predict power
  - For models with one slope only, the **standardized slope** (found using z-scored variables with $M = 0$ and $SD = 1$) is the same as Pearson's correlation → unambiguous "bivariate" effect size
  - For models with >1 slope, there are multiple potential measures of slope-specific effect size that you can choose from…

- Although **standardized slopes** are often used to index effect size in multiple-slope models, they **have problems** in some cases:
  - Ambiguous results for quadratic or multiplicative terms (z-scored product of 2 variables is not equal to product of 2 z-scored variables)
  - Differences in sample size across groups create different standardized slopes for categorical predictors given the same unstandardized mean difference (see Darlington & Hayes, 2016 ch. 8 for more)

Effect Size per Fixed Slope from t

- We can use t test-statistics to compute 2 different metrics of **partial effect sizes** (for slopes or their linear combinations)
  - Here "**partial**" refers to a slope's unique effect in models with multiple fixed slopes *(stay tuned for "semi-partial" alternatives)*
  - Why t-value? Effect sizes for fixed effect linear combinations, too
  - **Partial correlation r** (range is ± 1): $pr = \frac{t}{\sqrt{t^2 + DF_{den}}}$
    - Useful for quantitative predictors to convey strength of unique association for that slope
    - Can also get partial r from SAS PROC CORR, STATA PCORR, and pcor.test in R package ppcor
  - **Partial Cohen's d** (range is $\pm\infty$): $pd = \frac{2t}{\sqrt{DF_{den}}}$
    - Conveys difference between two groups in standard deviation units
    - Other common variants: Glass' delta uses SD for only 1 group; Hedges' g weights by the relative sample size in each group

From r to d:

$$d \approx \frac{2r}{\sqrt{1 + r^2}}$$

$$r \approx \frac{d}{\sqrt{4 + d^2}}$$

Effect Sizes for Our Example Results and Sample Sizes Needed for Power = .80

- **LvM Diff as $\boldsymbol{\beta_1}$**: $Est = 8.85, SE = 1.00, t(731) = 8.82, p < .001$
  - $r = \frac{8.82}{\sqrt{8.82^2 + 731}} = 0.31, d = \frac{2*8.82}{\sqrt{731}} = 0.65$ → ~per-group $\boldsymbol{n > 45}$

- **LvU Diff as $\boldsymbol{\beta_2}$**: $Est = 10.98, SE = 2.99, t(731) = 3.67, p < .001$
  - $r = \frac{3.67}{\sqrt{3.67^2 + 731}} = 0.13, d = \frac{2*3.67}{\sqrt{731}} = 0.27$ → ~ per-group $\boldsymbol{n > 175}$

- **MvU Diff as**: $\boldsymbol{\beta_2 - \beta_1}$: $Est = 2.13, SE = 3.03, t(731) = 0.70, p = .482$
  - $r = \frac{0.70}{\sqrt{0.70^2 + 731}} = 0.03, d = \frac{2*0.70}{\sqrt{731}} = 0.05$ → ~ per-group $\boldsymbol{n > 2,102}$

- **Model $\boldsymbol{R^2}$** $= .103, r = .322$ → ~overall $\boldsymbol{N > 85}$

Intermediate Summary

- For GLMs with **<u>one fixed slope</u>**, the significance test for that fixed slope is the same as the significance test for the model
  - Slope $\boldsymbol{\beta}_{unstd}$: $t = \frac{Est - H_0}{SE}$, $\boldsymbol{\beta}_{std}$ = Pearson r
  - Model: $F = t^2$, $R^2 = r^2$ because predicted $\hat{\boldsymbol{y}}_{\boldsymbol{i}}$ only uses $\boldsymbol{\beta}_{unstd}$

- For GLMs with **<u>2+ fixed slopes</u>**, the significance tests for those fixed slopes (or any linear combinations thereof) are NOT the same as the significance test for the overall model
  - Single test of <u>one fixed slope</u> via $\boldsymbol{t}$ (or z) → "Univariate Wald Test"
  - Joint test of <u>2+ fixed slopes</u> via $\boldsymbol{F}$ (or χ^2) → "Multivariate Wald Test"
    - F test-statistic is used to test the significance of the model $\boldsymbol{R}^2$ (the square of the $\boldsymbol{r}$ between model-predicted $\hat{\boldsymbol{y}}_{\boldsymbol{i}}$ and actual $\boldsymbol{y}_{\boldsymbol{i}}$, which is necessary whenever the predicted $\hat{\boldsymbol{y}}_{\boldsymbol{i}}$ uses multiple $\boldsymbol{\beta}_{unstd}$ slopes)
    - F test-statistic evaluates model R^2 *per DF spent to get it and DF leftover*

Nonlinear Trends of Quantitative Predictors

- Besides predictors with 3+ categories, another situation in which a single predictor variable may require more than one fixed slope to create its model prediction (its "effect" or "trend") is **when a quantitative predictor has a nonlinear relation with the outcome**

- We will examine <u>three types of examples</u> of this scenario:
  - **Curvilinear effect** of a quantitative predictor
    - Combine linear and quadratic slopes to create U-shape curve
    - Use natural-log transformed predictor to create an exponential curve
  - **Piecewise effects** for "sections" of a quantitative predictor
    - Also known as "linear splines" but each slope can be nonlinear, too
  - **Testing the assumption of linearity**: that equal differences between predictor values create equal outcome differences
    - Relevant for ordinal variables in which numbers are really just labels
    - Relevant for count predictors in which "more" may mean different things at different predictor values (e.g., "if and how much" predictors)

Curvilinear Trends of Quantitative Predictors

- The effect of a quantitative predictor does NOT have to be linear—curvilinear effects may be more theoretically reasonable or fit better
- There are many kinds of nonlinear trends—here are two examples:
  - **Quadratic (i.e., U-shaped)**: created by combining two predictors
    - "Linear": what it means when you enter the predictor by itself
    - "Quadratic": from also entering the predictor^2 (multiplied by itself)
    - Good to create relationships that **change directions**
    - Example for quadratic trend of $\boldsymbol{x_i}$: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{x_i}) + \boldsymbol{\beta_2}(\boldsymbol{x_i})^2 + \boldsymbol{e_i}$
  - **Exponential(ish)**: created from one nonlinearly-transformed predictor
    - Predictor = natural-log transform of predictor (positive values only)
    - Good to create relationships that look like **diminishing returns**
    - Example for exponential(ish) trend of $\boldsymbol{x_i}$: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{Log}[\boldsymbol{x_i}]) + \boldsymbol{e_i}$

How to Interpret Quadratic Slopes

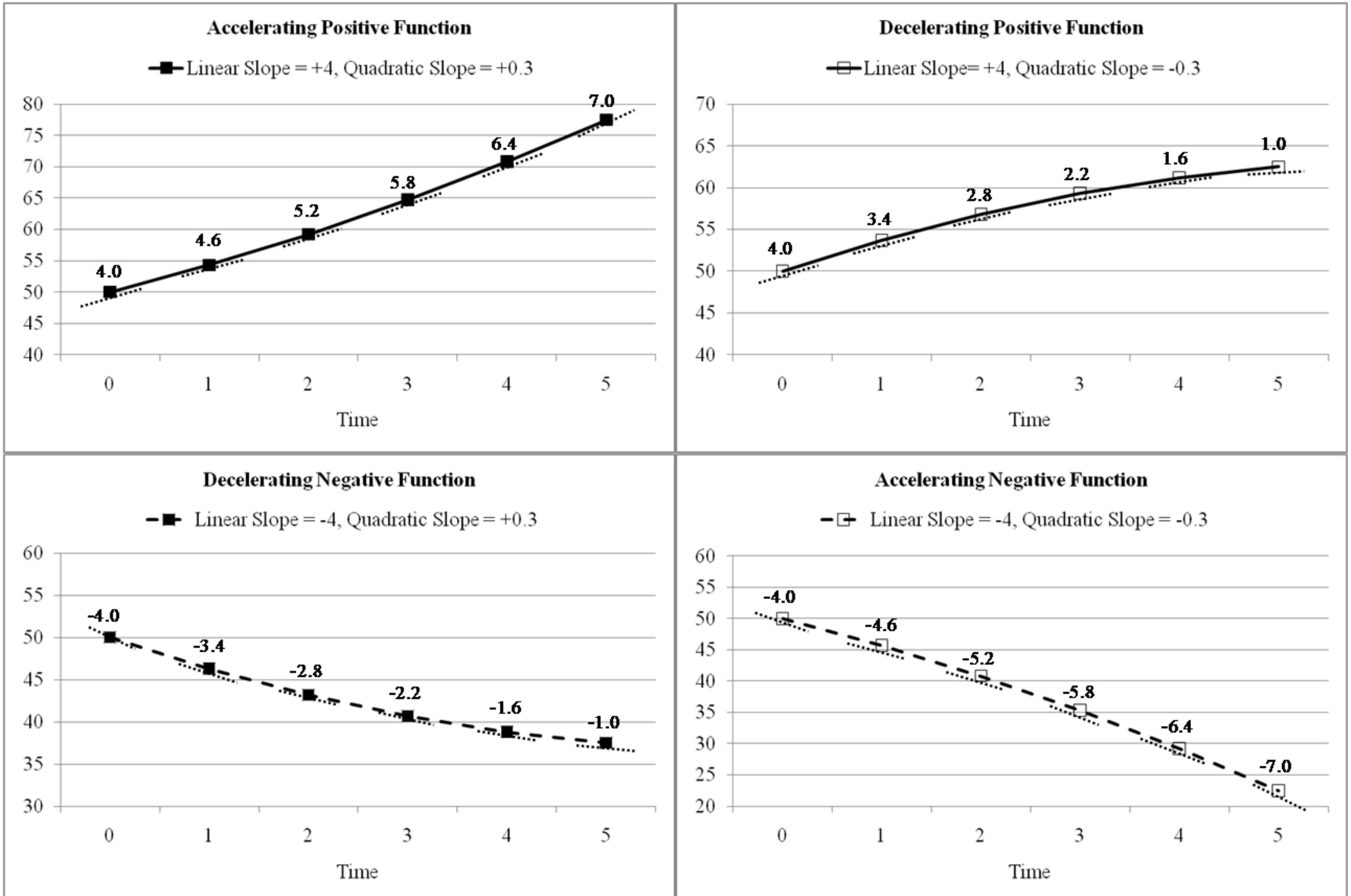
- A quadratic slope makes the effect of x_i change across itself!
  - Related to the ideas of position, velocity, and acceleration in physics
- Quadratic slope = HALF the rate of acceleration/deceleration
  - So to describe how the linear slope for x_i changes per unit difference in x_i, you must **multiply the quadratic slope for $\boldsymbol{x_i}$ by 2**
- If fixed linear slope = 4 at $x_i = 0$, with quadratic slope = 0.3?
  - "Instantaneous" linear rate of change is 4.0 at $x_i = 0$, is 4.6 at $x_i = 1$…
  - Btw: The "twice" rule comes from the derivatives of the function with respect to x_i:

Intercept (Position) at $x_i = X$: $\hat{y}_X = 50.0 + 4.0x_i + 0.3x_i^2$

First Derivative (Velocity) at X: $\dfrac{d\hat{y}_X}{d(X)} = 4.0 + 0.6x_i$

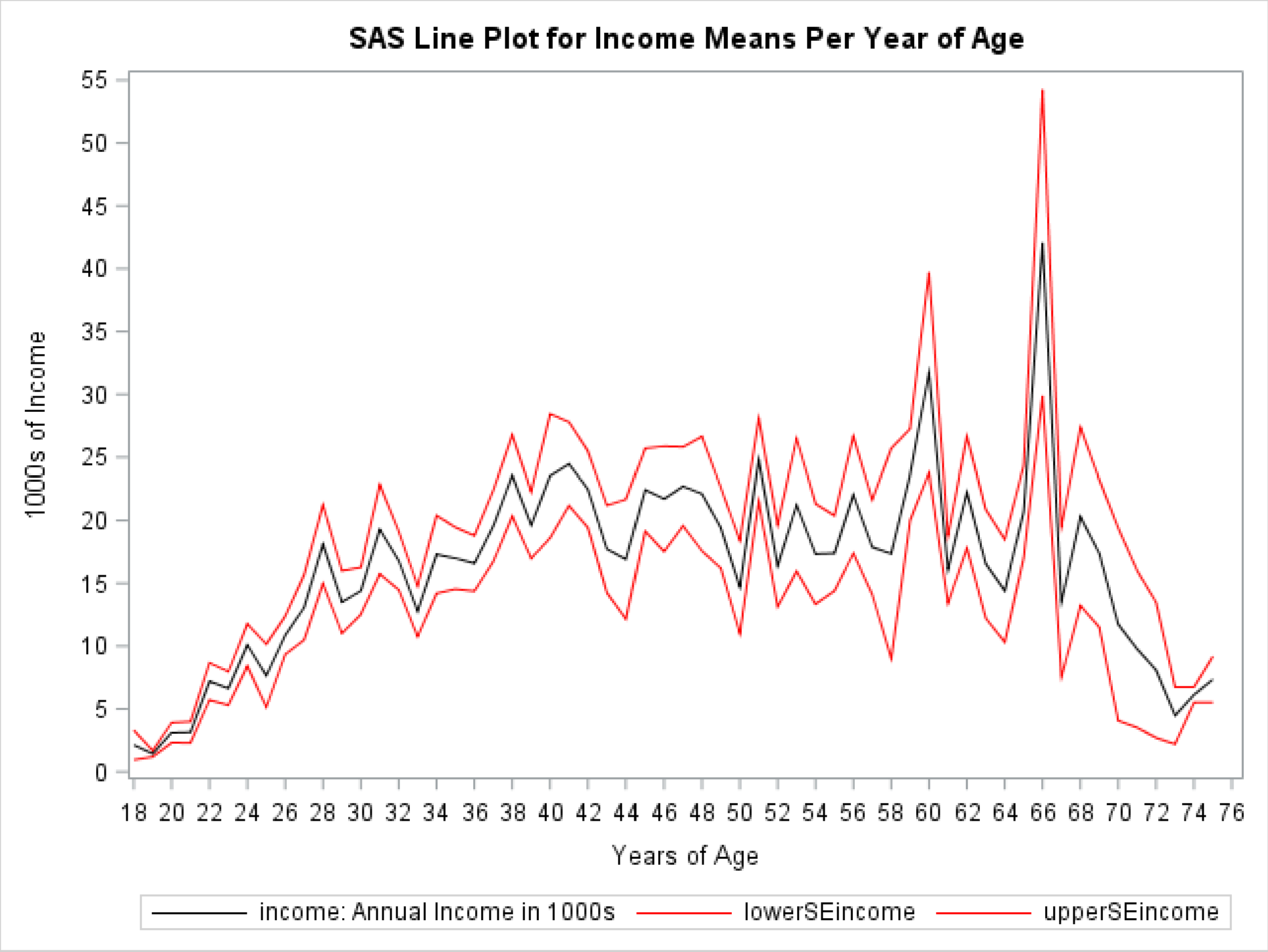
Second Derivative (Acceleration) at X: $\dfrac{d^2\hat{y}_X}{d(X)} = 0.6$

Quadratic Trends: Example of x_i = Time



Quadratic Trend for Age: GSS Example

- Black line = mean for each year of age; red lines = ±1 SE of mean
- Although noisy, this plot shows a clear quadratic function of age in predicting annual income (yay middle age!)

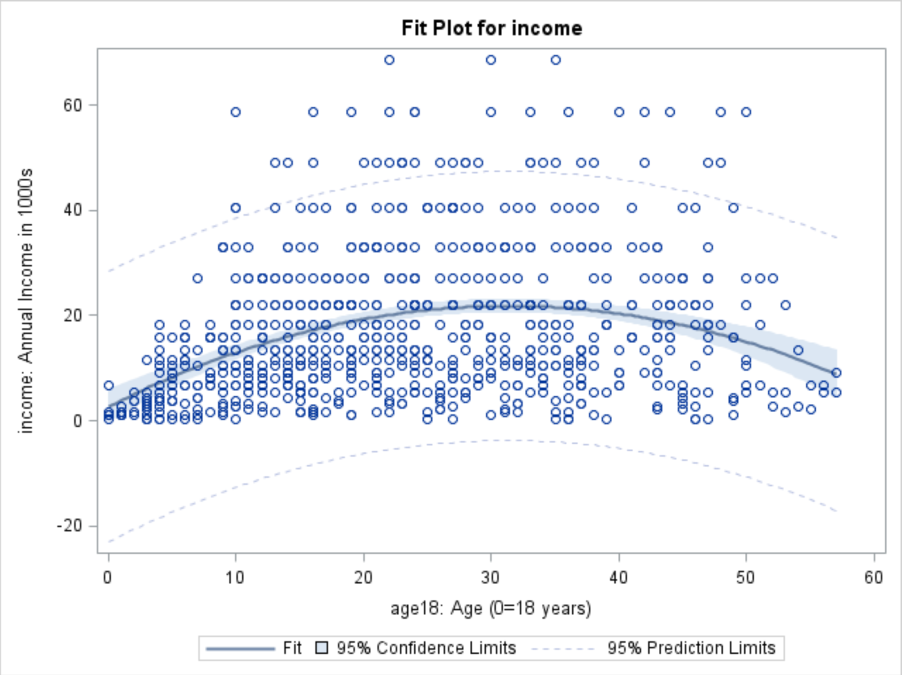


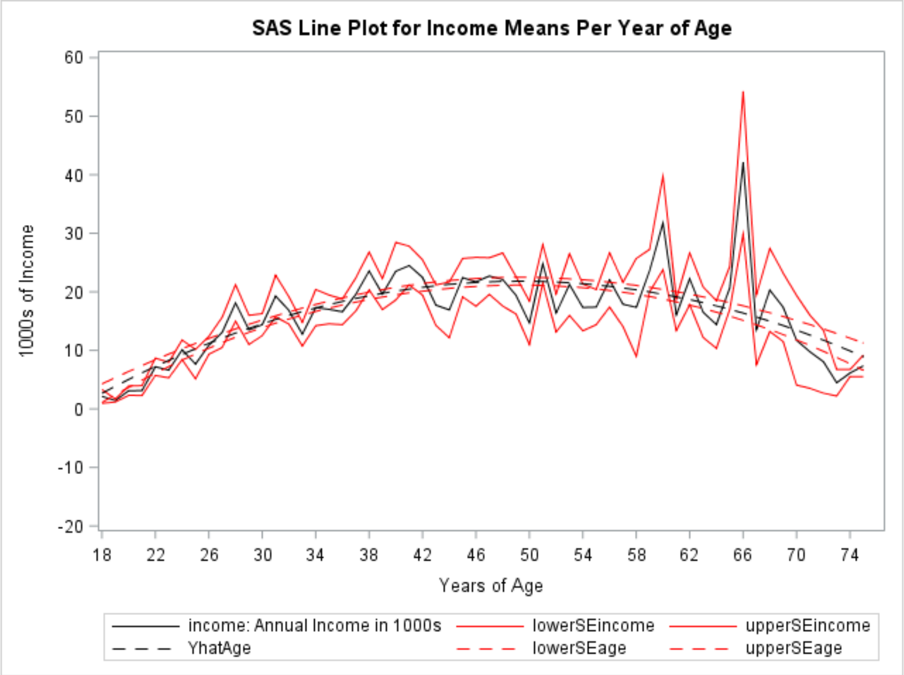
- Let's see what happens when we fit **a quadratic effect of age** (centered at 18, the minimum age) predicting annual income…

Quadratic Trend for Age: GSS Example

- $\boldsymbol{Income_i = \beta_0 + \beta_1(Age_i - 18) + \beta_2(Age_i - 18)^2 + e_i}$
  - **Intercept:** $\boldsymbol{\beta_0}$ = expected income at age 18 → $Est = 2.677, SE = 1.584, p < .001$
  - **Linear Age Slope:** $\boldsymbol{\beta_1}$ = instantaneous rate of change (or difference, actually) in income per year of age **at age = 18** → $Est = 1.223, SE = 0.135, p < .001$
  - **Quadratic Age Slope:** $\boldsymbol{\beta_2}$ = half the rate of acceleration (or deceleration here) per year of age **at any age** → $Est = -0.020, SE = 0.003, p < .001$
- Predicted income at other ages via linear combinations of fixed effects:
  - Age 30: $\hat{y}_{x=30} = \mathbf{2.677} + \mathbf{1.223}(\mathbf{12}) - \mathbf{0.020}(\mathbf{12})^2 = 14.540, SE = 0.647$
  - Age 50: $\hat{y}_{x=50} = \mathbf{2.677} + \mathbf{1.223}(\mathbf{32}) - \mathbf{0.020}(\mathbf{32})^2 = 21.809, SE = 0.668$
  - Age 70: $\hat{y}_{x=70} = \mathbf{2.677} + \mathbf{1.223}(\mathbf{52}) - \mathbf{0.020}(\mathbf{52})^2 = 13.448, SE = 1.659$
- Predicted linear age slope at other ages via linear combinations:
  - Age 30: $\widehat{\beta_1}_{x=30} = \mathbf{1.223} - \mathbf{0.020}(\mathbf{2} * \mathbf{12}) = 0.754, SE = 0.079$
  - Age 50: $\widehat{\beta_1}_{x=50} = \mathbf{1.223} - \mathbf{0.020}(\mathbf{2} * \mathbf{32}) = -0.027, SE = 0.047$
  - Age 70: $\beta_{1\,x=70} = \mathbf{1.223} - \mathbf{0.020}(\mathbf{2} * \mathbf{52}) = -0.809, SE = 0.135$
- Predicted age at maximum income (linear age slope = 0): $\frac{-\beta_1}{2 * \beta_2} + 18 = 48.575$

Quadratic Trend for Age: GSS Example

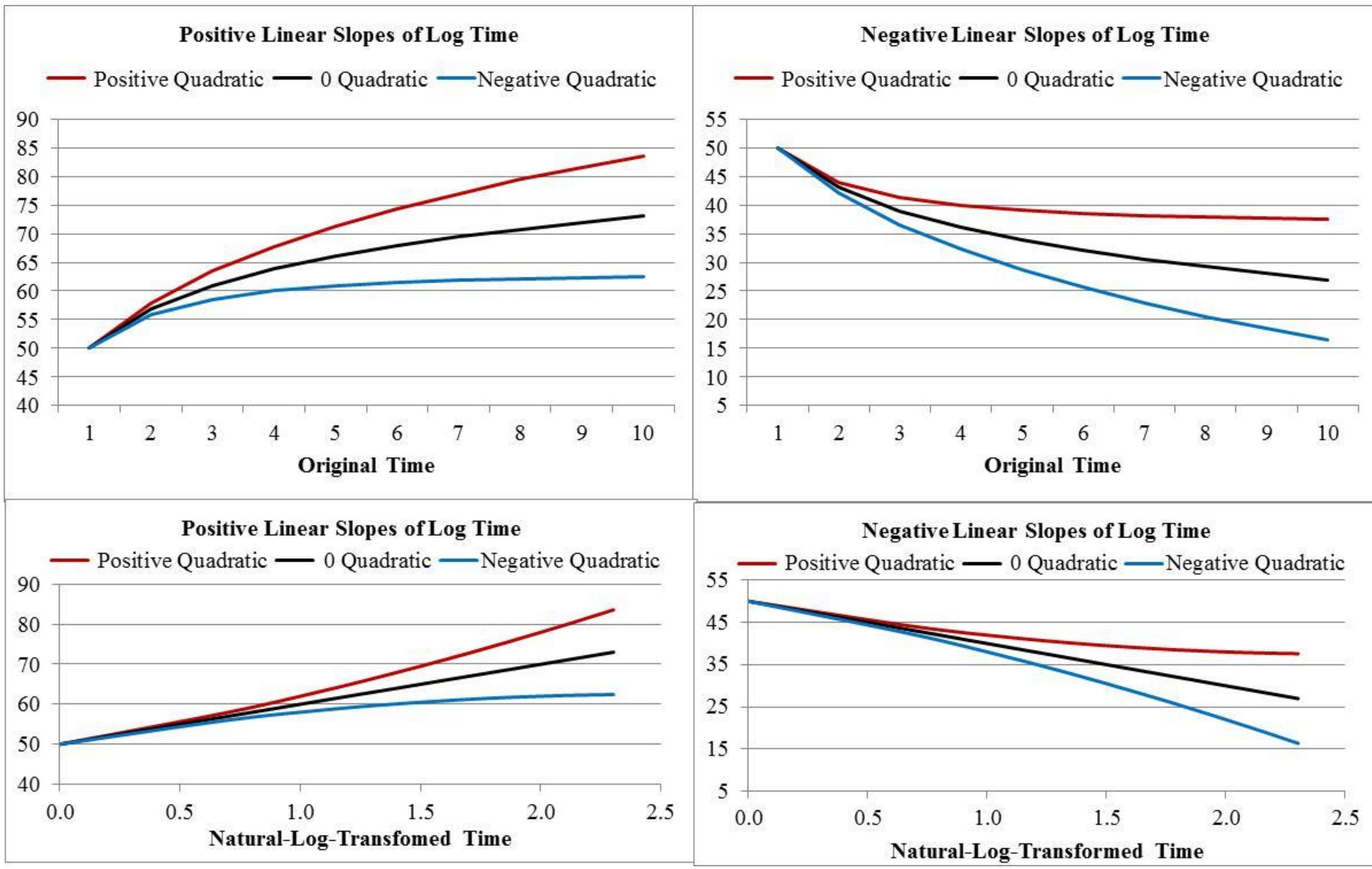




- Left: predicted regression line over individual scatterplot
  - From: $\mathbf{2.677 + 1.223(\mathit{Age_i} - 18) - 0.020(\mathit{Age_i} - 18)^2}$
- Right: predicted regression line over mean per age
  - $F(2,\ 731) = 47.00, MSE = 169.00, p < .001, R^2 = .114\ (r = .338)$
    - Since age and age² work together, I'd use model r as effect size

Exponential Trends: Example of x_i = Time

- A linear slope of log x_i (black lines) mimics an exponential trend across *original* x_i; adding a quadratic slope of log x_i (red or blue lines) can speed up or slow down the exponential(ish) trend

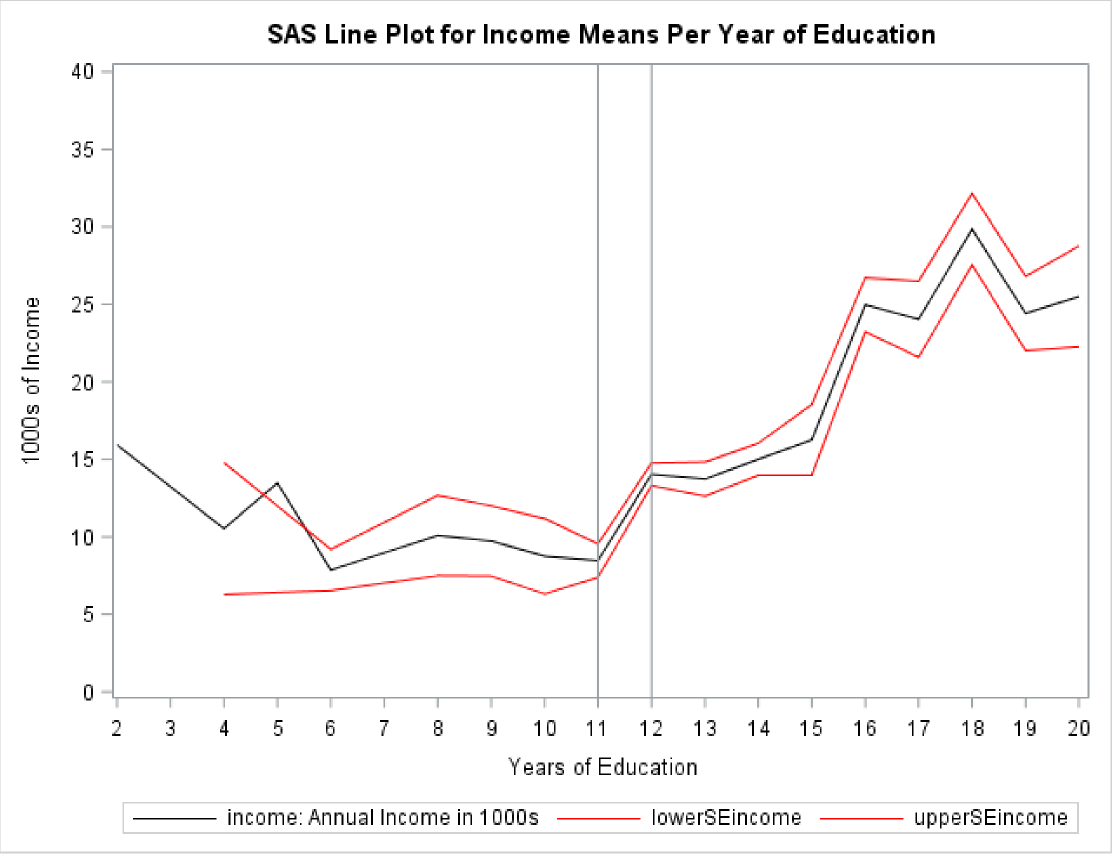


There is an **exponential** effect of **original x_i** on the outcome

There is a **linear** effect of **log x_i** on the outcome

Piecewise Slopes: GSS Example

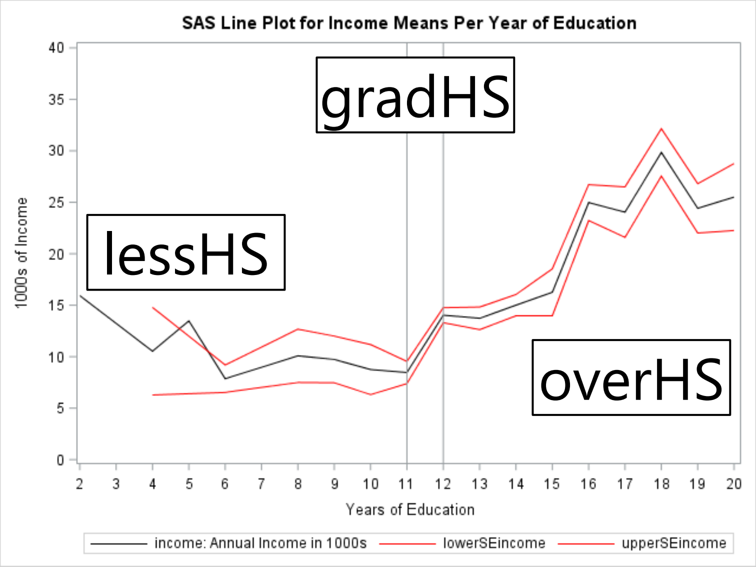
- What if the effect of "more education" varies across education? For example, I hypothesize for predicted annual income:
  - **Less than HS degree?** No effect of educ
  - **Get HS degree?** Acute "bump" relative to less than HS degree
  - **More than HS degree?** Positive effect of more educ (likely nonlinear)
- Plot: black line shows mean per year of educ, red lines show ± 1 SE



Piecewise Slopes Coding: GSS Example

Years Educ (x)	lessHS: Slope if x <12	gradHS: HS Grad? (0=no, 1=yes)	overHS: Slope if x >12
9	-2	0	0
10	-1	0	0
11 (int)	0	0	0
12	0	1	0
13	0	1	1
14	0	1	2
15	0	1	3
16	0	1	4
17	0	1	5
18	0	1	6

- Intercept = grade 11 (when all slopes = 0)
- **3 predictors** for educ:
  - **lessHS**: from grade 2 to 11
  - **gradHS**: acute bump for 12+
  - **overHS**: after grade 12 (to 20)



Piecewise Slopes: GSS Results

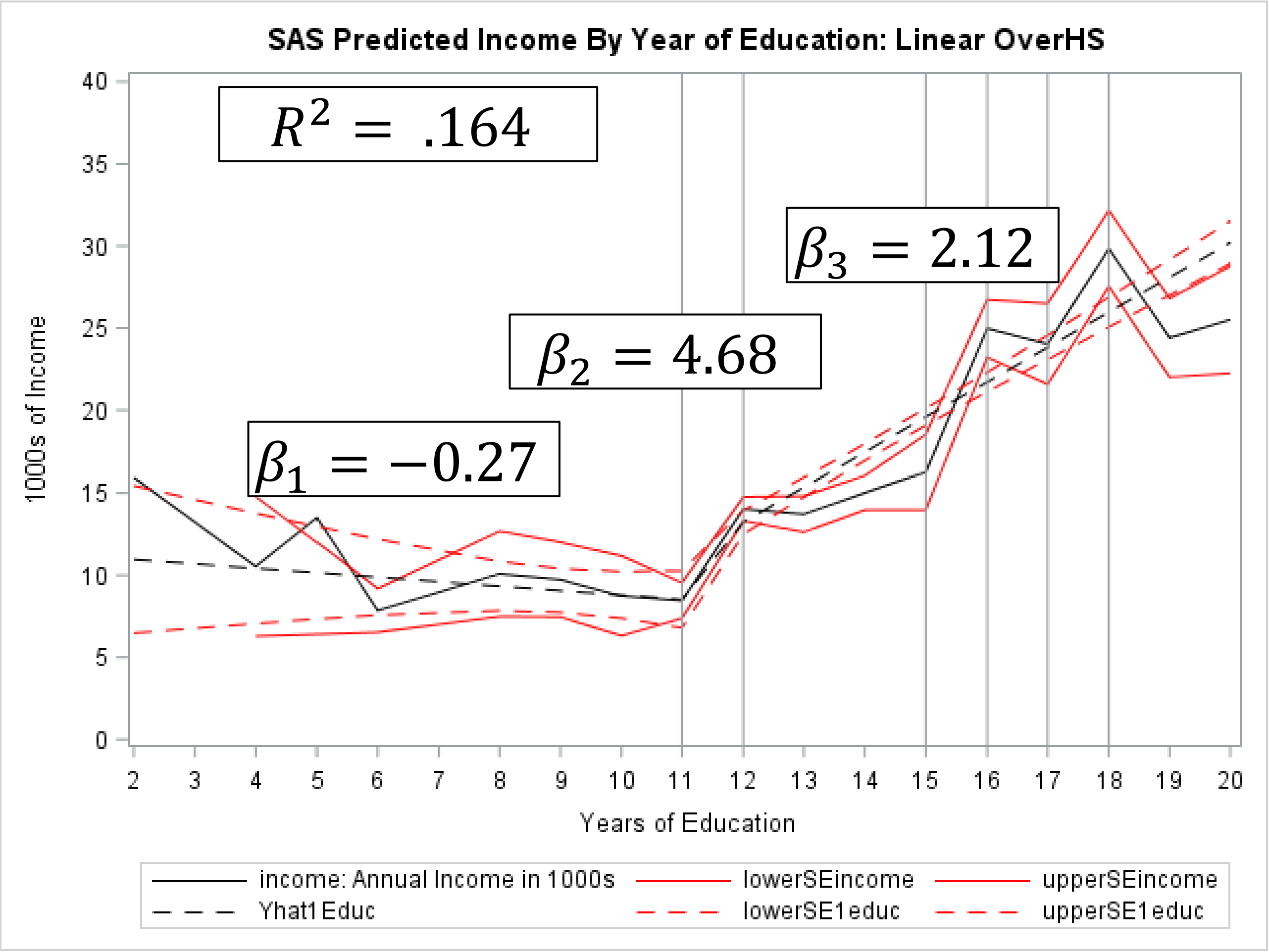
- After putting all three slopes in the model at the same time:

$$y_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(lessHS_i) + \boldsymbol{\beta_2}(gradHS_i) + \boldsymbol{\beta_3}(overHS_i) + \boldsymbol{e_i}$$

- Model: $F(3,\ 730) = 47.84, MSE = 159.61, p < .001, R^2 = .164\ (r = .404)$
  - $r = .404$ is effect size for overall prediction by education (three slopes)
- $\boldsymbol{\beta_0}$ = expected income when all predictors = 0 → 11 years of ed here
  - $Est = 8.53, SE = 1.73$ *(significance and effect size not relevant)*
- $\boldsymbol{\beta_1}$ = slope for difference in income <u>per year education</u> from 2 to 11 years
  - $Est = -0.27, SE = 0.60, t(730) = 0.65, p = .654, pr = -.017$
- $\boldsymbol{\beta_2}$ = acute difference (jump) in income between educ=11 and educ=12
  - $Est = 4.68, SE = 1.88, t(730) = 2.05, p = .013, pr = .092$
- $\boldsymbol{\beta_3}$ = slope for difference in income <u>per year education</u> from 12 to 20 years
  - $Est = 2.12, SE = 0.214, t(730) = 9.94, p < .001, pr = .345$

Piecewise Slopes: Linear Past 12 Years Ed?

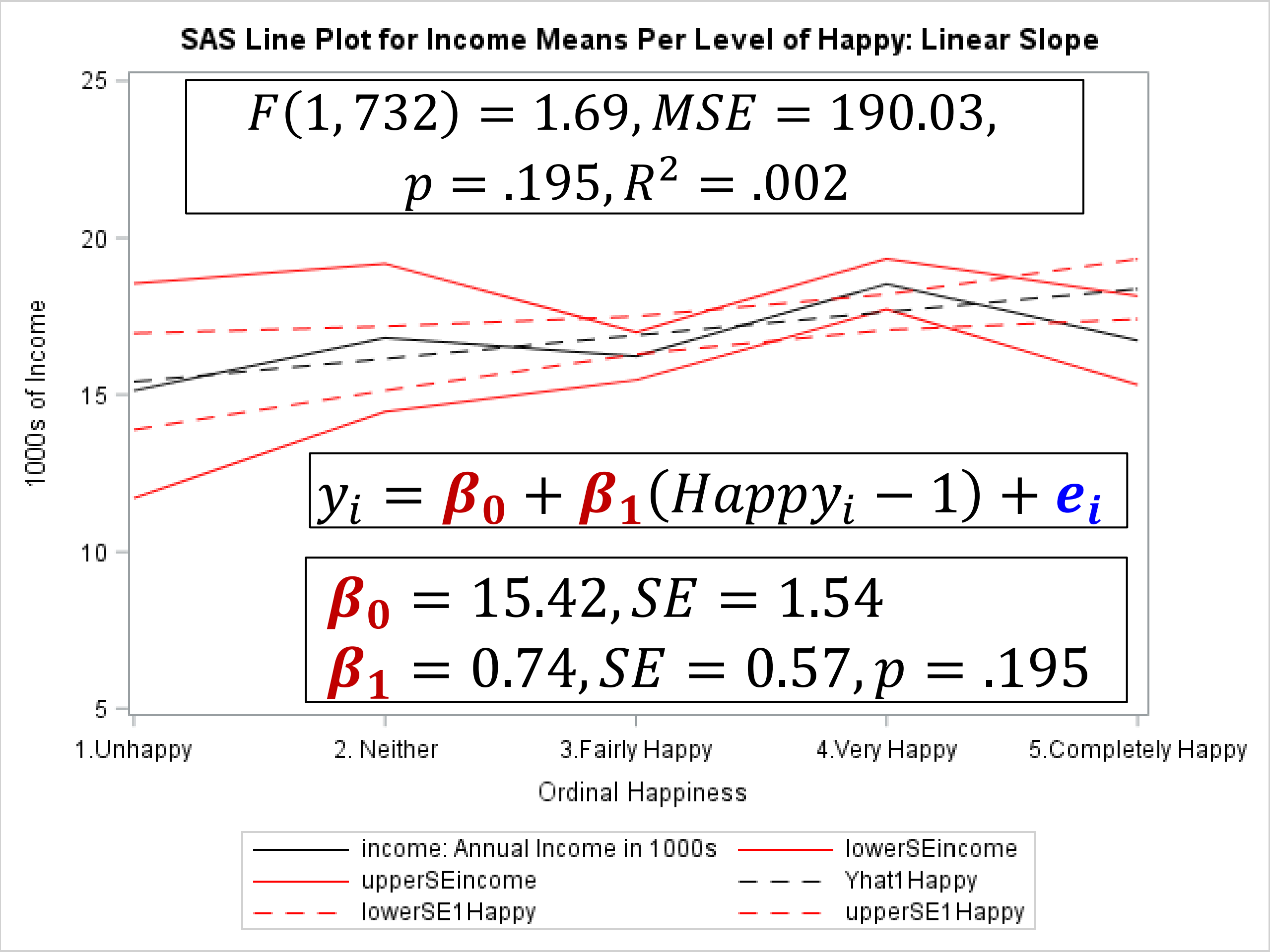
- The model (dashed lines) appears to capture the mean trend (solid lines) pretty well until 12 years of education…
- I think we need even more piecewise slopes after ed=12!
  - From 12 to 15
  - From 15 to 17-18
  - From 17-18 to 20



A Linear Slope for an Ordinal Predictor???

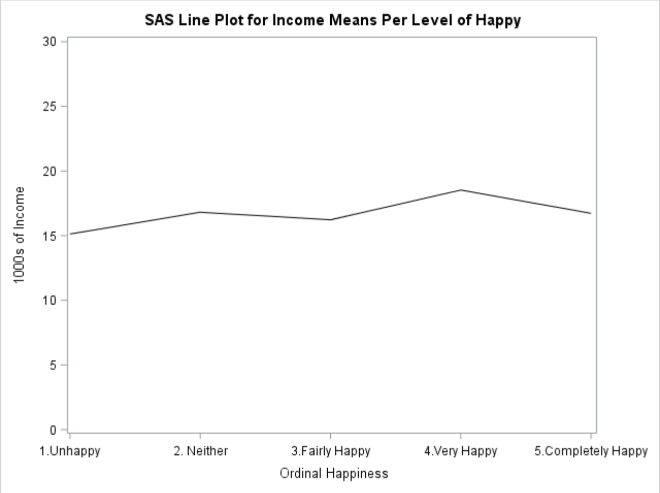
- Ordinal predictors with 5+ categories are often treated as interval by fitting a single linear slope for their overall effect ☹

- We can test this interval assumption by comparing the outcome differences between adjacent predictor values
  - Here: need 4 slopes, 1 for each transition between categories
  - Use "**sequential dummy coding**" to treat the predictor as "categorical" → 5 fixed effects used to distinguish each of 5 categories



Sequential Slopes for an Ordinal Predictor

Happy (x)	h1v2: Dif from 1 to 2	h2v3: Dif from 2 to 3	h3v4: Dif from 3 to 4	h4v5: Dif from 4 to 5
1 (int)	0	0	0	0
2	**1**	0	0	0
3	1	**1**	0	0
4	1	1	**1**	0
5	1	1	1	**1**

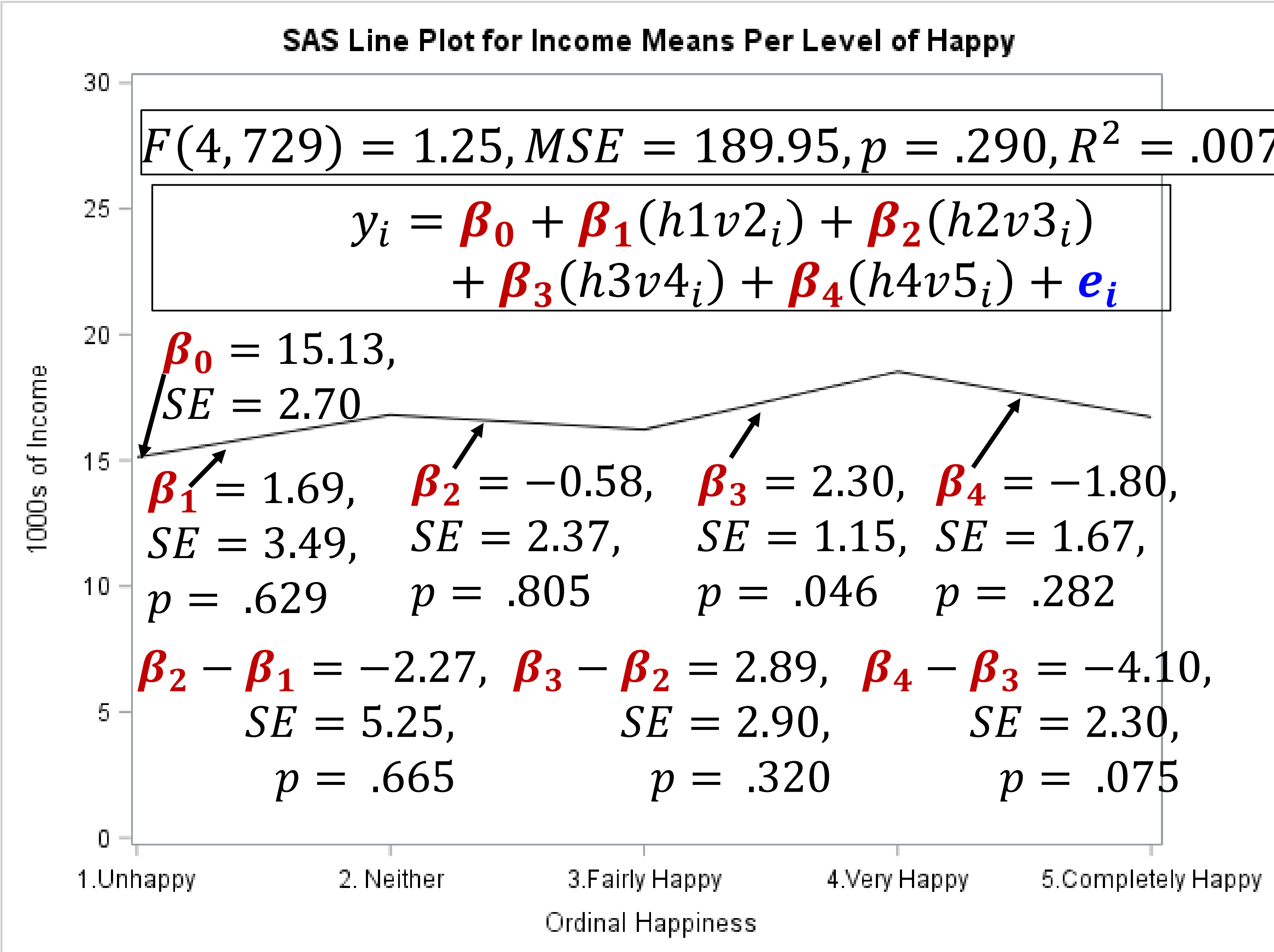


- Happy = 1 is where all slopes are 0, so it is the reference category (→ model intercept)
- The 4 slopes capture each adjacent category difference because each stays at **1** when done
  - Right: In indicator coding, the LvsM slope went back to 0, so the second slope is NOT successive (i.e., it reflects LvsU, not MvsU)

Group	LvM: Diff for Low vs Mid	LvU: Diff for Low vs Upp
Low	0	0
Mid	**1**	0
Upp	0	**1**

See p. 278 of: Darlington, R. B., & Hayes, A. F. (2016). *Regression analysis and linear models: Concepts, applications, and implementation*. Guilford.

Results from Testing Slope Differences



$F(4,729) = 1.25, MSE = 189.95, p = .290, R^2 = .007$

$$y_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(h1v2_i) + \boldsymbol{\beta_2}(h2v3_i) + \boldsymbol{\beta_3}(h3v4_i) + \boldsymbol{\beta_4}(h4v5_i) + \boldsymbol{e_i}$$

$\boldsymbol{\beta_0} = 15.13,$ $SE = 2.70$

$\boldsymbol{\beta_1} = 1.69,$ $SE = 3.49,$ $p = .629$

$\boldsymbol{\beta_2} = -0.58,$ $SE = 2.37,$ $p = .805$

$\boldsymbol{\beta_3} = 2.30,$ $SE = 1.15,$ $p = .046$

$\boldsymbol{\beta_4} = -1.80,$ $SE = 1.67,$ $p = .282$

$\boldsymbol{\beta_2} - \boldsymbol{\beta_1} = -2.27,$ $SE = 5.25,$ $p = .665$

$\boldsymbol{\beta_3} - \boldsymbol{\beta_2} = 2.89,$ $SE = 2.90,$ $p = .320$

$\boldsymbol{\beta_4} - \boldsymbol{\beta_3} = -4.10,$ $SE = 2.30,$ $p = .075$

Summary: Predictors with Multiple Fixed Slopes

- There are many scenarios in which a single predictor x_i needs **multiple fixed slopes** to describe its prediction of outcome y_i:
  - Predictor variables with $\boldsymbol{C}$ categories needs $\boldsymbol{C} - 1$ fixed slopes to distinguish its $\boldsymbol{C}$ possible different outcome means
    - "Indicator dummy coding" is useful for nominal predictors
    - "Sequential dummy coding" is useful for ordinal predictors
  - Should report significance and effect size for each mean difference of theoretical interest (not necessarily all possible differences, though)
  - Nonlinear effects of quantitative predictor variables (via quadratic or exponential curves; piecewise slopes or curves) may require 2+ slopes
    - Predictors work together to summarize overall "trend" of x_i (so effect size for each fixed slope may be less important than overall model R^2)

- We want to know the significance of **each** fixed slope (via univariate Wald test of $(Est - H_0)/SE$ via t test-statistic) as well as significance of the **model $\boldsymbol{R^2}$** (as multivariate Wald test via F test-statistic)
  - Model R^2 = squared Pearson $\boldsymbol{r}$ between predicted $\boldsymbol{\hat{y}_i}$ and actual $\boldsymbol{y_i}$