Example 4: Fixed and Random Slopes of Level-1 Predictors and Cross-Level Interactions in General Multilevel Models for Two-Level Nested Outcomes

(complete syntax and output available for STATA, R, and SAS electronically)

This example uses real data from a math test given at the end of 10th grade in a midwestern rectangular state. These analyses include 13,802 students from 94 schools, with 31–515 students in each school (M = 139). We will **use "hybrid" models** to examine how student lunch status (0 = pay full price for lunch, 1= receive reduced lunch, 2= receive free lunch) predicts student math test scores. We will use **sequential indicator coding** to first distinguish paid lunch from reduced or free lunch, and then further distinguish reduced lunch from free lunch.

STATA Syntax for Importing and Preparing Data for Analysis:

```
// Define global variable for file location to be replaced in code below
// \\Client\ precedes path in Virtual Desktop outside H drive
   global filesave "C:\Dropbox\23_PSQF6272\PSQF6272_Example4"

// Open trimmed example excel data file from sheet "grade10" and clear away existing data
   clear // clear memory in case of open data
   import excel "$filesave\Example4_Data.xlsx", firstrow case(preserve) sheet("grade10")
   clear

// Add labels to original variables
   label variable districtID "districtID: District ID number"
   label variable studentID  "studentID: Student ID number"
   label variable schoolID   "schoolID: School ID number"
   label variable lunch      "lunch: 0=Paid, 1=Reduced, 2=Free"
   label variable math       "math: Math Test Score"

display "STATA Descriptive Statistics within Student-Level Data"
tabulate lunch
```

Descriptive Statistics in Student-Level Data (from SAS):

lunch: 0=Paid, 1=Reduced, 2=Free

lunch	Frequency	Percent	Cumulative Frequency	Cumulative Percent
0	9059	69.25	9059	69.25
1	1140	8.71	10199	77.96
2	2883	22.04	13082	100.00

Given that the lunch predictor is ordinal, we will use **sequential coding** to distinguish each transition:

		PvRF	PRvF
Lunch=0	Paid	0	0
Lunch=1	Reduced	1	0
Lunch=2	Free	1	1

```
// Create new dummy-coded predictors (start as blank, then recode)
   gen PvRF=.
   gen PRvF=.
   replace PvRF=0 if lunch==0  // Replace each for paid lunch
   replace PRvF=0 if lunch==0
   replace PvRF=1 if lunch==1  // Replace each for reduced lunch
   replace PRvF=0 if lunch==1
   replace PvRF=1 if lunch==2  // Replace each for free lunch
   replace PRvF=1 if lunch==2
   label variable PvRF "PvRF: 0=Paid, 1=Reduced or Free Lunch"
   label variable PRvF "PRvF: 0=Paid or Reduced, 1=Free Lunch"

// Create indicator variables for demo purposes
   gen lunch0=.
   gen lunch1=.
   gen lunch2=.
   replace lunch0=1 if lunch==0  // Replace each for paid lunch
   replace lunch1=0 if lunch==0
   replace lunch2=0 if lunch==0
```

```
    replace lunch0=0 if lunch==1  // Replace each for reduced lunch
    replace lunch1=1 if lunch==1
    replace lunch2=0 if lunch==1
    replace lunch0=0 if lunch==2  // Replace each for free lunch
    replace lunch1=0 if lunch==2
    replace lunch2=1 if lunch==2
    label variable lunch0 "lunch0: 1=Paid Lunch"
    label variable lunch1 "lunch1: 1=Reduced Lunch"
    label variable lunch2 "lunch2: 1=Free Lunch"

// Filter to complete cases before computing cluster means
    egen nmiss=rowmiss(math lunch)
    drop if nmiss>0

// Compute cluster means for level-1 variables
    sort schoolID
    egen schoolN = count(math),    by(schoolID)
    egen CM_math = mean(math),     by(schoolID)
    egen CM_PvRF = mean(PvRF),     by(schoolID)
    egen CM_PRvF = mean(PRvF),     by(schoolID)
    egen CM_lunch0 = mean(lunch0), by(schoolID)
    egen CM_lunch1 = mean(lunch1), by(schoolID)
    egen CM_lunch2 = mean(lunch2), by(schoolID)

display "STATA Descriptive Statistics within School-Level Data"
preserve  // Save for later use, then compute school-level dataset
collapse  schoolN CM_math CM_PvRF CM_PRvF CM_lunch0 CM_lunch1 CM_lunch2, by(schoolID)
format    schoolN CM_math CM_PvRF CM_PRvF CM_lunch0 CM_lunch1 CM_lunch2  %4.2f
summarize schoolN CM_math CM_PvRF CM_PRvF CM_lunch0 CM_lunch1 CM_lunch2, format
```

Descriptive Statistics in School-Level Data (from SAS):

Variable and Label	N	Mean	Std Dev	Minimum	Maximum
schoolN: # Students Sampled Per School	94	139.17	138.20	31.00	515.00
CM_math: School Mean Math Test Score	94	47.73	6.97	29.45	61.61
CM_PvRF: School Mean 0=No, 1=Reduced or Free Lunch	94	0.30	0.21	0.00	0.80
CM_PRvF: School Mean 0=Paid or Reduced, 1=Free Lunch	94	0.19	0.16	0.00	0.68
CM_lunch0: School Mean Paid Lunch	94	0.70	0.21	0.20	1.00
CM_lunch1: School Mean Reduced Lunch	94	0.11	0.13	0.00	0.73
CM_lunch2: School Mean Free Lunch	94	0.19	0.16	0.00	0.68

```
// Go back to student-level dataset
    restore
// Center cluster means so ref school has 0=.70, 1=.11, and 2=.19
    gen CM_PvRF30 = CM_PvRF - .30
    gen CM_PRvF19 = CM_PRvF - .19
// Cluster-mean-center level-1 predictors for random slopes
    gen WC_PvRF = PvRF - CM_PvRF
    gen WC_PRvF = PRvF - CM_PRvF
    label variable WC_PvRF "WC_PvRF: Within-Cluster Paid vs Reduced/Free Lunch"
    label variable WC_PRvF "WC_PRvF: Within-Cluster Paid/Reduced vs Free Lunch"

display "STATA Descriptive Statistics within Student-Level Data"
summarize WC_PvRF WC_PRvF, detail
```

Descriptive Statistics in Student-Level Data (from SAS):

Variable	N	Mean	Variance	Std Dev	Minimum	Maximum
WC_PvRF	13082	-0.000	**0.164**	0.405	-0.803	0.989
WC_PRvF	13082	-0.000	**0.140**	0.374	-0.678	0.989

These predictor variances will be used in computing slope reliability later…

R Syntax for Importing and Preparing Data for Analysis (after loading packages *readxl*, *TeachingDemos*, *psych*, *lme4*, *lmerTest*, *performance*, and *ordinal*):

```
# Define variables for working directory and data name -- CHANGE THESE
filesave = "C:\\Dropbox/23_PSQF6272/PSQF6272_Example4/"
filename = "Example4_Data.xlsx"
setwd(dir=filesave)

# Load Jonathan's custom R functions from folder within working directory
functions = paste0("R functions/",dir("R functions/"))
temp = lapply(X=functions, FUN=source)
#  Import trimmed example excel data file from sheet "grade10"
Example4 = read_excel(paste0(filesave,filename), sheet="grade10")
# Convert to data frame to use in analysis
Example4 = as.data.frame(Example4)

print("R Descriptive Statistics within Student-Level Data")
prop.table(table(x=Example4$lunch, useNA="ifany"))

# Create new dummy-coded predictors (start as blank, then recode)
Example4$PvRF=NA; Example4$PRvF=NA
Example4$PvRF[which(Example4$lunch==0)]=0  # Replace each for paid lunch
Example4$PRvF[which(Example4$lunch==0)]=0
Example4$PvRF[which(Example4$lunch==1)]=1  # Replace each for reduced
Example4$PRvF[which(Example4$lunch==1)]=0
Example4$PvRF[which(Example4$lunch==2)]=1  # Replace each for free lunch
Example4$PRvF[which(Example4$lunch==2)]=1

# Create indicator variables for demo purposes
Example4$lunch0=NA; Example4$lunch1=NA; Example4$lunch2=NA
Example4$lunch0[which(Example4$lunch==0)]=1  # Replace each for paid lunch
Example4$lunch1[which(Example4$lunch==0)]=0
Example4$lunch2[which(Example4$lunch==0)]=0
Example4$lunch0[which(Example4$lunch==1)]=0  # Replace each for reduced
Example4$lunch1[which(Example4$lunch==1)]=1
Example4$lunch2[which(Example4$lunch==1)]=0
Example4$lunch0[which(Example4$lunch==2)]=0  # Replace each for free lunch
Example4$lunch1[which(Example4$lunch==2)]=0
Example4$lunch2[which(Example4$lunch==2)]=1

# Filter to only cases complete on all variables to be used below (before cluster means)
Example4 = Example4[complete.cases(Example4[ , c("math","lunch")]),]

# Compute cluster means for level-1 variables using Jonathan's function
Example4 = addUnitMeans(data=Example4, unitVariable="schoolID",
                        meanVariables=c("math","PvRF","PRvF","lunch0","lunch1","lunch2"),
           newNames=c("CM_math","CM_PvRF","CM_PRvF","CM_lunch0","CM_lunch1","CM_lunch2"))

print("R Descriptive Statistics within School-Level Data")
schoolMeans = unique(Example4[,c("schoolID","NperschoolID","CM_math","CM_PvRF",
                                 "CM_PRvF","CM_lunch0","CM_lunch1","CM_lunch2")])
describe(x=schoolMeans[ , c("NperschoolID","CM_math","CM_PvRF","CM_PRvF",
                            "CM_lunch0","CM_lunch1","CM_lunch2")])

# Center cluster means so ref school has 0=.70, 1=.11, and 2=.19
Example4$CM_PvRF30 = Example4$CM_PvRF - .30
Example4$CM_PRvF19 = Example4$CM_PRvF - .19
# Cluster-mean-center level-1 predictors for random slopes
Example4$WC_PvRF = Example4$PvRF - Example4$CM_PvRF
Example4$WC_PRvF = Example4$PRvF - Example4$CM_PRvF
# WC_PvRF= "WC_PvRF: Within-Cluster Paid vs Reduced/Free Lunch"
# WC_PRvF= "WC_PRvF: Within-Cluster Paid/Reduced vs Free Lunch"

print("R Descriptive Statistics within Student-Level Data")
describe(x=Example4[ , c("WC_PvRF","WC_PRvF")])
```

Model 1: Empty Means, Random Intercept Model for the Math Outcome

Level 1: $Math_{pc} = \beta_{0c} + e_{pc}$
Level 2: $\beta_{0c} = \gamma_{00} + U_{0c}$

```
display "STATA Model 1: Empty Means, Random Intercept for Math"
mixed math , || schoolID: , reml dfmethod(satterthwaite) dftable(pvalue) nolog

------------------------------------------------------------------
        math |      Coef.   Std. Err.           DF       t    P>|t|
-------------+----------------------------------------------------
       _cons |   47.75583   .7230123          92.8    66.05   0.000
------------------------------------------------------------------
------------------------------------------------------------------------------
  Random-effects Parameters  |   Estimate   Std. Err.     [95% Conf. Interval]
-----------------------------+------------------------------------------------
schoolID: Identity           |
                  var(_cons) |   45.45287   7.153705      33.3882    61.87705
-----------------------------+------------------------------------------------
               var(Residual) |   253.1759   3.141547     247.0928    259.4087
------------------------------------------------------------------------------
LR test vs. linear model: chibar2(01) = 1860.21       Prob >= chibar2 = 0.0000

display "-2LL = " e(ll)*-2      // Print -2LL for model
-2LL = 109789.72

estat icc                       // Intraclass correlation
------------------------------------------------------------------------------
                       Level |        ICC   Std. Err.     [95% Conf. Interval]
-----------------------------+------------------------------------------------
                    schoolID |   .1522053   .0203803      .1164008    .1965729
------------------------------------------------------------------------------
```

```
print("R Model 1: Empty Means, Random Intercept for Math")
Model1 = lmer(data=Example4, REML=TRUE, formula=math~1+(1|schoolID))
print("Show results using Satterthwaite DDF including -2LL as deviance")
llikAIC(Model1, chkREML=FALSE); summary(Model1, ddf="Satterthwaite")

'log Lik.' -54894.858 (df=3) → LL for model

       AIC        BIC     logLik   deviance   df.resid
109795.717 109818.154 -54894.858 109789.717  13079.000  → deviance = −2LL for model

Random effects:
 Groups   Name        Variance Std.Dev.
 schoolID (Intercept)  45.453   6.7419
 Residual             253.176  15.9115
Number of obs: 13082, groups:  schoolID, 94

Fixed effects:
            Estimate Std. Error       df t value  Pr(>|t|)
(Intercept) 47.75583    0.72301 93.90282   66.051 < 2.2e-16

print("Show intraclass correlation and its LRT")
icc(Model1); ranova(Model1)

# Intraclass Correlation Coefficient
    Adjusted ICC: 0.152
  Unadjusted ICC: 0.152
```

$$\text{ICC} = \frac{\tau^2_{U_0}}{\tau^2_{U_0} + \sigma^2_e} = \frac{45.453}{45.453 + 253.176} = \mathbf{.152}$$

```
ANOVA-like table for random-effects: Single term deletions
               npar   logLik    AIC     LRT Df Pr(>Chisq)
<none>            3 -54894.9 109796
(1 | schoolID)    2 -55825.0 111654 1860.21  1 < 2.22e-16
```

Likelihood Ratio Test (LRT) Statistic:
$= -2(54894.9 + 55825.0) = \mathbf{1860.21}$

Design effect using mean #students per school: = 1 + ((L1*n* – 1) * ICC) → 1 + [(139−1)*.152] = **22.03**
Effective sample size: $N_{effective}$ = (#Total Obs) / Design Effect → 13,082 / 22.03 = **594!!!**

Random intercept reliability: $$\text{ICC2} = \frac{\tau^2_{U_0}}{\tau^2_{U_0} + \sigma^2_e / L1n} = \frac{45.453}{45.453 + 253.176/139} = \mathbf{.961}$$

95% random intercept CI: Fixed effect ± 1.96*SQRT(random variance)
47.756 ± 1.96*SQRT(45.453) = 34.55 to 60.96
→ 95% of our sample's schools are predicted to have school mean math from 34.542 to 60.970

Empty Model for the Level-1 Student Ordinal Lunch Predictor

```
display "STATA: Empty Means, Random Intercept for Ordinal Lunch"
display "Provides submodel thresholds (multiply by -1 to get intercepts)"
meologit lunch , || schoolID: , nolog
display "-2LL = " e(ll)*-2        // Print -2LL for model
display "ICC = " 1.793843/(1.793843+3.29)
```

```
------------------------------------------------------------------------------
       lunch |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       /cut1 |   1.203654   .1428788     8.42   0.000     .9236164    1.483691
       /cut2 |   1.767587   .1433791    12.33   0.000     1.486569    2.048605
-------------+----------------------------------------------------------------
schoolID     |
   var(_cons)|   1.793843   .3054923                      1.284768    2.504635
------------------------------------------------------------------------------
LR test vs. ologit model: chibar2(01) = 2981.76       Prob >= chibar2 = 0.0000
```

```
print("R: Empty Means, Random Intercept for Ordinal Lunch using ordinal package")
EmptyLunch = clmm(data=Example4, link="logit", formula=as.factor(lunch)~1+(1|schoolID))
print("Show -2LL, provides submodel thresholds (multiply by -1 to get intercepts)")
-2*logLik(EmptyLunch); summary(EmptyLunch)

'log Lik.' 17961.429 (df=3) → -2LL for model

link  threshold nobs  logLik   AIC      niter   max.grad cond.H
 logit flexible  13082 -8980.71 17967.43 96(664) 1.32e-04 3.2e+02

Random effects:
 Groups   Name        Variance Std.Dev.
 schoolID (Intercept) 1.783    1.3353

Threshold coefficients:
    Estimate Std. Error z value
0|1  1.20318    0.14275  8.4285
1|2  1.76711    0.14325 12.3358

print("ICC using pi^2/3 = 3.29 as L1 residual variance"); icc(EmptyLunch)
# Intraclass Correlation Coefficient
    Adjusted ICC: 0.351
  Unadjusted ICC: 0.351

print("R: Single-level empty model predicting Ordinal Lunch ignoring school")
SingleLunch = clm(data=Example4, link="logit", formula=as.factor(lunch)~1)
print("Likelihood Ratio Test for Addition of Random Intercept Variance")
DevTest=-2*(logLik(SingleLunch)-logLik(EmptyLunch))
Pvalue=pchisq((DevTest), df=1, lower.tail=FALSE)
print("Test Statistic and P-values for DF=1");
DevTest; 'log Lik.' 2980.7553 (df=2)
Pvalue 'log Lik.' 0 (df=2)
```

Model 2a: Add Level-1 Binary Student Paid vs Reduced/Free Lunch

Level 1: $Math_{pc} = \beta_{0c} + \beta_{1c}(PaidvRedFree_{pc}) + e_{pc}$

Level 2: $\beta_{0c} = \gamma_{00} + U_{0c}$

$\beta_{1c} = \gamma_{10}$

```
display "STATA Model 2a: Add L1 Binary Student Paid vs Reduced/Free Lunch"
mixed math c.PvRF, || schoolID: , reml dfmethod(satterthwaite) dftable(pvalue) nolog
display "-2LL = " e(ll)*-2      // Print -2LL for model
```

```
print("R Model 2a: Add L1 Binary Student Paid vs Reduced/Free Lunch")
Model2a = lmer(data=Example4, REML=TRUE, formula=math~1+PvRF+(1|schoolID))
print("Show results using Satterthwaite DDF including -2LL as deviance")
llikAIC(Model2a, chkREML=FALSE); summary(Model2a, ddf="Satterthwaite")
```

```
       AIC        BIC     logLik   deviance   df.resid
109023.794 109053.710 -54507.897 109015.794  13078.000

Random effects:
 Groups   Name        Variance Std.Dev.
 schoolID (Intercept)  27.225   5.2178
 Residual             239.347  15.4708
Number of obs: 13082, groups:  schoolID, 94

Fixed effects:
               Estimate  Std. Error         df t value  Pr(>|t|)
(Intercept)    50.61478     0.57974    96.88213  87.306 < 2.2e-16
PvRF           -9.42877     0.33181 12946.48431 -28.416 < 2.2e-16
```

Intercept γ_{00} =

PvRF γ_{10} =

```
print("Psuedo-R2 relative to empty means model using Jonathan's function")
pseudoRSquaredinator(smallerModel=Model1, largerModel=Model2a)

R2 Random.(Intercept)          R2 L1.sigma2
          0.401021463           0.054622347
```

Pseudo-R2 Relative to CovEmpty (from SAS)

Name	CovParm	Subject	Estimate	StdErr	PseudoR2
CovEmpty	UN(1,1)	schoolID	45.3682	7.1288	.
CovEmpty	Residual		253.18	3.1416	.
CovSmush	UN(1,1)	schoolID	27.2239	4.5119	0.39993
CovSmush	Residual		239.35	2.9703	0.05463

What does this pattern of explained variance at each level tell us about the level-1 slope?

Model 2b: Let's Fix It—Add L2 Cluster Mean of Paid vs Reduced/Free Lunch

Level 1: $Math_{pc} = \beta_{0c} + \beta_{1c}\left(PaidvRedFree_{pc}\right) + e_{pc}$

Level 2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMPaidvRedFree_{c} - .30) + U_{0c}$

$\beta_{1c} = \gamma_{10}$

Remember:
L2 between = L1 within + L2 contextual
L2 between = $\gamma_{10} + \gamma_{01}$

```
display "STATA Model 2b: Add L2 Cluster Mean of Paid vs Reduced/Free Lunch"
mixed math c.PvRF c.CM_PvRF30, || schoolID: , ///
           reml dfmethod(satterthwaite) dftable(pvalue) nolog
display "-2LL = " e(ll)*-2        // Print -2LL for model
lincom c.PvRF*1 + c.CM_PvRF30*1   // L2 PvRF Between Slope
estimates store Fix               // Save fit for LRT
```

```
print("R Model 2b: Add L2 Cluster Mean of Paid vs Reduced/Free Lunch")
Model2b = lmer(data=Example4, REML=TRUE, formula=math~1+PvRF+CM_PvRF30+(1|schoolID))
print("Show results using Satterthwaite DDF including -2LL as deviance")
llikAIC(Model2b, chkREML=FALSE); summary(Model2b, ddf="Satterthwaite")
```

```
      AIC        BIC     logLik   deviance   df.resid
108971.850 109009.245 -54480.925 108961.850  13077.000

Random effects:
 Groups   Name        Variance Std.Dev.
 schoolID (Intercept)  13.89    3.7269
 Residual             239.41   15.4730

Fixed effects:
               Estimate  Std. Error          df  t value          Pr(>|t|)
(Intercept)    50.60344     0.43920    91.73106 115.2185         < 2.2e-16
PvRF           -9.17288     0.33443 12979.90061 -27.4288         < 2.2e-16
CM_PvRF30     -16.84334     2.02514    83.28613  -8.3171 0.000000000001483
```

Results from previous Model 2a:

```
              Estimate   Std. Error
(Intercept)   50.61478      0.57974
PvRF          -9.42877      0.33181
```

Intercept γ_{00} =

PvRF: γ_{10} =

CM_PvRF30: γ_{01} =

```
print("L2 PvRF Between Slope"); contest1D(Model2b, L=c(0,1,1))
    Estimate Std. Error        df     t value     Pr(>|t|)
1 -26.016225  1.9973365 78.802728 -13.025459 2.4112541e-21
```

```
print("Psuedo-R2 relative to empty model using Jonathan's function")
pseudoRSquaredinator(smallerModel=Model1, largerModel=Model2b)
R2 Random.(Intercept)          R2 L1.sigma2
          0.694414340           0.054364134
```

Pseudo-R2 Relative to CovEmpty (from SAS)
Change in Pseudo-R2 for CovSmush vs. CovContext

Name	CovParm	Subject	Estimate	StdErr	PseudoR2	PseudoR2Change
CovEmpty	UN(1,1)	schoolID	45.3682	7.1288	.	.
CovEmpty	Residual		253.18	3.1416	.	.
CovSmush	UN(1,1)	schoolID	27.2239	4.5119	0.39993	.
CovSmush	Residual		239.35	2.9703	0.05463	.
CovContext	UN(1,1)	schoolID	13.8884	2.6315	0.69387	0.29394
CovContext	Residual		239.41	2.9718	0.05438	-0.00026

Which effect is responsible for each reduction in variance (relative to the empty model, and then across models 2a and 2b)?

Model 2c: Add Random Slope for <u>Cluster-Mean-Centered</u> Paid vs Reduced/Free Lunch

Level 1: $Math_{pc} = \beta_{0c} + \beta_{1c}\left(PaidvRedFree_{pc}\right)$
$$+ \beta_{2c}\left(PaidvRedFree_{pc} - CMPaidvRedFree_{c}\right) + e_{pc}$$

Level 2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMPaidvRedFree_{c} - .30) + U_{0c}$
$$\beta_{1c} = \gamma_{10}; \quad \beta_{2c} = U_{2c}$$

```
display "STATA Model 2c: Add Random Slope for Cluster-MC Paid vs Reduced/Free Lunch"
mixed math c.PvRF c.CM_PvRF30, || schoolID: WC_PvRF, ///
          cov(un) reml dfmethod(satterthwaite) dftable(pvalue) nolog

                                                F(2,     99.21)     =     211.64
Log restricted-likelihood = -54435.942          Prob > F            =     0.0000
------------------------------------------------------------------
        math |      Coef.   Std. Err.          DF       t    P>|t|
-------------+----------------------------------------------------
        PvRF |  -8.438614   .5841238         71.7   -14.45   0.000
   CM_PvRF30 |  -16.49722   2.027635        110.5    -8.14   0.000
       _cons |   50.35261   .5109494         89.9    98.55   0.000
------------------------------------------------------------------
------------------------------------------------------------------------------
  Random-effects Parameters  |   Estimate   Std. Err.     [95% Conf. Interval]
-----------------------------+------------------------------------------------
schoolID: Unstructured       |
                var(WC_PvRF) |   14.21053   3.809487      8.402808    24.03234
                  var(_cons) |   13.80124   2.619506      9.513888    20.02064
          cov(WC_PvRF,_cons) |  -7.320018   2.675847     -12.56458   -2.075454
-----------------------------+------------------------------------------------
               var(Residual) |   236.7918   2.945818      231.0879    242.6365
------------------------------------------------------------------------------
LR test vs. linear model: chi2(3) = 448.62                Prob > chi2 = 0.0000
```

The LRT above is for the entire matrix of random effect variances and covariances at once, which is not helpful in testing the addition of the random slope variance (and the covariance with the random intercept). Instead, we will ask for a custom LRT below using the saved results from the previous model against the current below.

```
display "-2LL = " e(ll)*-2      // Print -2LL for model
-2LL = 108871.88

estat recovariance, relevel(schoolID) correlation  // Random effect correlations

Random-effects correlation matrix for level schoolID
             |   WC_PvRF      _cons
-------------+----------------------
     WC_PvRF |         1
       _cons | -.5226946          1

estimates store Rand            // Save fit for LRT
lrtest Rand Fix                 // LRT for random slope

Likelihood-ratio test                                 LR chi2(2)  =     89.97
(Assumption: Fix nested in Rand)                      Prob > chi2 =    0.0000

Note: The reported degrees of freedom assumes the null hypothesis is not on the boundary of
the parameter space.  If this is not true, then the reported test is conservative.
```

Translation: if you are testing a variance that must be > 0, use a mixture χ^2 distribution (with df = 1,2) instead.

```
Note: LR tests based on REML are valid only when the fixed-effects specification is
identical for both models.
```

Translation: Keep the same fixed effects when testing new random effect variances and covariances in REML!

```
print("R Model 2c: Add Random Slope for Cluster-MC Paid vs Reduced/Free Lunch")
Model2c = lmer(data=Example4, REML=TRUE,
               formula=math~1+PvRF+CM_PvRF30+(1+WC_PvRF|schoolID))
print("Show results using Satterthwaite DDF including -2LL as deviance")
llikAIC(Model2c, chkREML=FALSE); summary(Model2c, ddf="Satterthwaite")

       AIC        BIC      logLik   deviance   df.resid
108885.885 108938.238 -54435.942 108871.885  13075.000

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 schoolID (Intercept)  13.800   3.7149
          WC_PvRF      14.209   3.7694  -0.523
 Residual             236.792  15.3880

Fixed effects:
             Estimate Std. Error       df  t value          Pr(>|t|)
(Intercept)  50.35262    0.51093 83.77456  98.5501         < 2.2e-16
PvRF         -8.43864    0.58410 87.08263 -14.4472         < 2.2e-16
CM_PvRF30   -16.49718    2.02757 96.77574  -8.1364 0.000000000001394
```

Results from fixed slope Model 2b:

	Estimate	Std. Error
(Intercept)	50.60344	0.43920
PvRF	**-9.17288**	**0.33443**
CM_PvRF30	-16.84334	2.02514

```
print("LRT for random slope"); ranova(Model2c)
                                   npar  logLik    AIC     LRT Df Pr(>Chisq)
<none>                                7 -54435.9 108886
WC_PvRF in (1 + WC_PvRF | schoolID)   5 -54480.9 108972 89.9652  2 < 2.22e-16
```

95% random slope CI: Fixed effect ± 1.96*SQRT(random variance)

−8.439 ± 1.96*SQRT(14.209) = −15.827 to −1.050

→ 95% of our sample's schools are predicted to have a disadvantage in math for students with reduced or free lunch (relative to paid lunch) from 1.050 to 15.827!

Random slope reliability: $SR = \frac{\tau^2_{U_1}}{\tau^2_{U_1} + [\sigma^2_e/(L1n * var)]} = \frac{14.209}{14.209 + [236.792/(139 * 0.164)]} = \mathbf{.578}$

Btw, 0.164 is the variance of the cluster-mean-centered L1 WC_PvRF that has the random slope.

Recap: We already knew that schools differed from each other in their mean math scores, but now we know that schools also differ from each other in the disadvantage related to receiving reduced or free lunch. Next, we try to predict those school slope differences by school lunch composition—does the lunch-related disadvantage depend on how many students are also disadvantaged in your school?

Model 3a: Add Intra-Variable Cross-Level Interaction (Keeping Random Slope for PvRF)

Level 1: $Math_{pc} = \beta_{0c} + \beta_{1c}\left(PaidvRedFree_{pc}\right) + \beta_{2c}\left(PaidvRedFree_{pc} - CMPaidvRedFree_c\right) + e_{pc}$

Level 2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMPaidvRedFree_c - .30) + U_{0c}$

$\beta_{1c} = \gamma_{10} + \gamma_{11}(CMPaidvRedFree_c - .30);\quad \beta_{2c} = U_{2c}$

```
display "STATA Model 3a: Add Intra-Variable Cross-Level Interaction"
mixed math c.PvRF c.CM_PvRF30 c.PvRF#c.CM_PvRF30, || schoolID: WC_PvRF, ///
           cov(un) reml dfmethod(satterthwaite) dftable(pvalue) nolog
display "-2LL = " e(ll)*-2      // Print -2LL for model

print("R Model 3a: Add Intra-Variable Cross-Level Interaction")
Model3a = lmer(data=Example4, REML=TRUE, formula=math~1+PvRF+CM_PvRF30
               +PvRF:CM_PvRF30+(1+WC_PvRF|schoolID))
print("Show results using Satterthwaite DDF including -2LL as deviance")
llikAIC(Model3a, chkREML=FALSE); summary(Model3a, ddf="Satterthwaite")
```

```
       AIC         BIC      logLik    deviance    df.resid
108881.965 108941.797 -54432.982 108865.965  13074.000

Random effects:
 Groups    Name         Variance Std.Dev. Corr
 schoolID (Intercept)  14.014   3.7436
          WC_PvRF      13.360   3.6551  -0.513
 Residual             236.783  15.3878

Fixed effects:
                Estimate Std. Error        df  t value        Pr(>|t|)
(Intercept)     50.24354    0.51591  85.49280  97.3890       < 2.2e-16
PvRF            -8.67286    0.59280  99.99580 -14.6304       < 2.2e-16
CM_PvRF30      -18.76167    2.55594  85.23325  -7.3404 0.0000000001165
PvRF:CM_PvRF30   3.92284    2.64128 116.21113   1.4852          0.1402
```

Intercept γ_{00} =

PvRF: γ_{10} =

CM_PvRF30: γ_{01} =

PvRF*CM_PvRF30: γ_{11} =

```
print("Psuedo-R2 relative to random slope model using Jonathan's function")
pseudoRSquaredinator(smallerModel=Model2c, largerModel=Model3a)
```

```
R2 Random.(Intercept)      R2 Random.WC_PvRF         R2 L1.sigma2
      -0.015503972822       0.059740737707       0.000038342876
```

Pseudo-R2 Relative to CovRand (from SAS)

Name	CovParm	Subject	Estimate	StdErr	PseudoR2
CovRand	UN(1,1)	schoolID	13.7927	2.6170	.
CovRand	UN(2,2)	schoolID	14.2050	3.8075	.
CovRand	Residual		236.79	2.9458	.
Cov3a	UN(1,1)	schoolID	13.9932	2.6449	-0.014543
Cov3a	UN(2,2)	schoolID	13.3598	3.6857	0.059499
Cov3a	Residual		236.79	2.9456	0.000033

Which variance should have been reduced by the new cross-level interaction?

Model 3b: Let's Fix It—Add Level-2 Quadratic Interaction

Level 1: $Math_{pc} = \beta_{0c} + \beta_{1c}\left(PaidvRedFree_{pc}\right) + \beta_{2c}\left(PaidvRedFree_{pc} - CMPaidvRedFree_{c}\right) + e_{pc}$

Level 2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMPaidvRedFree_{c} - .30) + \gamma_{02}(CMPaidvRedFree_{c} - .30)^2 + U_{0c}$

$\beta_{1c} = \gamma_{10} + \gamma_{11}(CMPaidvRedFree_{c} - .30);\quad \beta_{2c} = U_{2c}$

Simple L2 PvRF Between Slope $= \gamma_{10} + \gamma_{01}$
L2 PvRF Between*CM_PvRF30 $= \gamma_{11} + \gamma_{02}$

```
display "STATA Model 3b: Add Level-2 Quadratic Interaction"
mixed math c.PvRF c.CM_PvRF30 c.PvRF#c.CM_PvRF30 c.CM_PvRF30#c.CM_PvRF30, ///
          || schoolID: WC_PvRF, cov(un) reml dfmethod(satterthwaite) dftable(pvalue) nolog
display "-2LL = " e(ll)*-2        // Print -2LL for model
lincom c.PvRF*1 + c.CM_PvRF30*1                           // Simple L2 PvRF Between Slope
lincom c.PvRF#c.CM_PvRF30*1 + c.CM_PvRF30#c.CM_PvRF30*1   // L2 PvRF Between*CM_PvRF30
predict pred3b
corr math pred3b                  // Get total r to make R2
display "Total-R2 = " r(rho)^2   // Print total R2 relative to empty model

print("R Model 3b: Add Level-2 Quadratic Interaction")
print("R re-ordered quadratic to come right after main effects")
Model3b = lmer(data=Example4, REML=TRUE, formula=math~1+PvRF+CM_PvRF30
               +I(CM_PvRF30^2) +PvRF:CM_PvRF30 +(1+WC_PvRF|schoolID))
print("Show results using Satterthwaite DDF including -2LL as deviance")
llikAIC(Model3b, chkREML=FALSE); summary(Model3b, ddf="Satterthwaite")

      AIC       BIC    logLik  deviance  df.resid
108876.08 108943.39 -54429.04 108858.08  13073.00

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 schoolID (Intercept)  13.790   3.7134
          WC_PvRF      13.192   3.6321  -0.480
 Residual             236.780  15.3877
```

Fixed effects from Model 3a:	Estimate	Std. Error
(Intercept)	50.24354	0.51591
PvRF	-8.67286	0.59280
CM_PvRF30	-18.76167	2.55594
PvRF:CM_PvRF30	**3.92284**	**2.64128**

```
Fixed effects:
                Estimate Std. Error        df  t value       Pr(>|t|)
(Intercept)     50.76302    0.65135  97.67129  77.9357        < 2.2e-16
PvRF            -8.80537    0.60063 100.38443 -14.6602        < 2.2e-16
CM_PvRF30      -18.06823    2.58668  88.62438  -6.9851 0.0000000005043
I(CM_PvRF30^2) -11.91241    9.36263 103.13257  -1.2723          0.2061
PvRF:CM_PvRF30   5.37916    2.87523  93.11701   1.8709          0.0645
```

Intercept γ_{00} =

PvRF: γ_{10} =

CM_PvRF30: γ_{01} =

PvRF*CM_PvRF30: γ_{11} =

CM_PvRF30^2: γ_{02} =

```
print("Simple PvRF L2 Between Slope"); contest1D(Model3b, L=c(0,1,1,0,0))
print("L2 PvRF Between*CM_PvRF30");    contest1D(Model3b, L=c(0,0,0,1,1))

    Estimate Std. Error        df     t value       Pr(>|t|)
1 -26.873605  2.5573197 90.371372 -10.508504 2.4067581e-17
1 -6.5332526   8.648862 79.146827 -0.7553887 0.45225796

print("Psuedo-R2 relative to random slope model using Jonathan's function")
pseudoRSquaredinator(smallerModel=Model2c, largerModel=Model3b)

R2 Random.(Intercept)     R2 Random.WC_PvRF          R2 L1.sigma2
       0.000772735293        0.071534856468       0.000050090621

print("Total-R2 relative to empty means model using Jonathan's function")
totalRSquaredinator(data=Example4, dvName="math", model=Model3b)
0.16409255
```

Pseudo-R2 Relative to CovRand (from SAS)
Change in Pseudo-R2 for Cov3a vs. Cov3b

Name	CovParm	Subject	Estimate	StdErr	PseudoR2	PseudoR2Change
CovRand	UN(1,1)	schoolID	13.7927	2.6170	.	.
CovRand	UN(2,2)	schoolID	14.2050	3.8075	.	.
CovRand	Residual		236.79	2.9458	.	.
Cov3a	UN(1,1)	schoolID	13.9932	2.6449	-0.014543	.
Cov3a	UN(2,2)	schoolID	13.3598	3.6857	0.059499	.
Cov3a	Residual		236.79	2.9456	0.000033	.
Cov3b	UN(1,1)	schoolID	13.7671	2.6236	0.001852	0.016396
Cov3b	UN(2,2)	schoolID	13.1962	3.6505	0.071022	0.011523
Cov3b	Residual		236.78	2.9456	0.000044	0.000011

What does this pattern of change in variance tell us about this model relative to the previous model?

Model 4a: Add 2 Main Effect Slopes for Paid/Reduced vs Free Lunch

Level 1: $Math_{pc} = \beta_{0c} + \beta_{1c}\left(PaidvRedFree_{pc}\right)$
$+ \beta_{2c}\left(PaidvRedFree_{pc} - CMPaidvRedFree_{c}\right)$
$+ \beta_{3c}\left(PaidRedvFree_{pc}\right) + e_{pc}$

Level 2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMPaidvRedFree_{c} - .30) + \gamma_{02}(CMPaidvRedFree_{c} - .30)^2$
$+ \gamma_{03}(CMPaidRedvFree_{c} - .19) + U_{0c}$
$\beta_{1c} = \gamma_{10} + \gamma_{11}(CMPaidvRedFree_{c} - .30);\quad \beta_{2c} = U_{2c}$
$\beta_{3c} = \gamma_{30}$

Simple L2 PvRF Between Slope $= \gamma_{10} + \gamma_{01}$
L2 PvRF Between*CM_PvRF30 $= \gamma_{11} + \gamma_{02}$
L2 PRvF Between Slope $= \gamma_{30} + \gamma_{03}$

```
display "STATA Model 4a: Add 2 Main Effect Slopes for Paid/Reduced vs Free Lunch"
mixed math c.PvRF c.CM_PvRF30 c.PvRF#c.CM_PvRF30 c.CM_PvRF30#c.CM_PvRF30 ///
           c.PRvF c.CM_PRvF19, || schoolID: WC_PvRF, ///
           cov(un) reml dfmethod(satterthwaite) dftable(pvalue) nolog
display "-2LL = " e(ll)*-2       // Print -2LL for model
estimates store Fix2             // Save fit for LRT
lincom c.PvRF*1 + c.CM_PvRF30*1                          // Simple L2 PvRF Between Slope
lincom c.PvRF#c.CM_PvRF30*1 + c.CM_PvRF30#c.CM_PvRF30*1  // L2 PvRF Between*CM_PvRF30
lincom c.PRvF*1 + c.CM_PRvF19*1                          // L2 PRvF Between Slope

print("R Model 4a: Add 2 Main Effect Slopes for Paid/Reduced vs Free Lunch")
print("R re-ordered new main effects and quadratic before interaction")
Model4a = lmer(data=Example4, REML=TRUE, formula=math~1+PvRF+CM_PvRF30+PRvF+CM_PRvF19
               +I(CM_PvRF30^2) +PvRF:CM_PvRF30 +(1+WC_PvRF|schoolID))
print("Show results using Satterthwaite DDF including -2LL as deviance")
llikAIC(Model4a, chkREML=FALSE); summary(Model4a, ddf="Satterthwaite")
```

```
Random effects:
 Groups   Name        Variance Std.Dev. Corr
 schoolID (Intercept)  13.934   3.7329
          WC_PvRF      12.221   3.4959  -0.477
 Residual             235.726  15.3534
```

```
Fixed effects from Model 3b:
                Estimate Std. Error
(Intercept)     50.76302    0.65135
PvRF            -8.80537    0.60063
CM_PvRF30      -18.06823    2.58668
I(CM_PvRF30^2) -11.91241    9.36263
PvRF:CM_PvRF30   5.37916    2.87523
```

```
Fixed effects:
                  Estimate  Std. Error         df t value  Pr(>|t|)
(Intercept)       50.69385     0.66263   97.58624 76.5036 < 2.2e-16
PvRF              -5.83385     0.70117  208.42506 -8.3202 1.143e-14
CM_PvRF30        -22.39799     3.98457  101.07772 -5.6212 1.687e-07
PRvF              -4.46526     0.56777 12452.84944 -7.8646 4.009e-15
CM_PRvF19          6.47672     4.57263   85.64499  1.4164   0.16028
I(CM_PvRF30^2)   -11.81434     9.64630   97.91842 -1.2248   0.22361
PvRF:CM_PvRF30     5.67466     2.82137   92.82405  2.0113   0.04719
```

Which fixed effects have changed in their interpretation after adding PRvF and CM_PRvF19?

```
print("Simple PvRF L2 Between Slope"); contest1D(Model4a, L=c(0,1,1,0,0,0,0))
print("L2 PvRF Between*CM_PvRF30");    contest1D(Model4a, L=c(0,0,0,0,0,1,1))
print("PRvF L2 Between Slope");        contest1D(Model4a, L=c(0,1,1,0,0,0,0))

    Estimate  Std. Error        df    t value            Pr(>|t|)
1 -28.231834   3.9425631 97.92529 -7.1607817  0.00000000014965018
1  -6.1396712  8.9362577 75.751034 -0.68705172 0.4941482
1   2.0114656  4.5129655 81.119399  0.44570818 0.65699536

print("Psuedo-R2 relative to random slope model using Jonathan's function")
pseudoRSquaredinator(smallerModel=Model2c, largerModel=Model4a)

R2 Random.(Intercept)     R2 Random.WC_PvRF          R2 L1.sigma2
        -0.0097030886          0.1398892194          0.0045032888

print("Total-R2 relative to empty means model using Jonathan's function")
totalRSquaredinator(data=Example4, dvName="math", model=Model4a)
0.16866537
```

Pseudo-R2 Relative to CovRand
Change in Pseudo-R2 for Cov3b vs. Cov4a

Name	CovParm	Subject	Estimate	StdErr	PseudoR2	PseudoR2Change
CovRand	UN(1,1)	schoolID	13.7927	2.6170	.	.
CovRand	UN(2,2)	schoolID	14.2050	3.8075	.	.
CovRand	Residual		236.79	2.9458	.	.
Cov3b	UN(1,1)	schoolID	13.7671	2.6236	0.00185	.
Cov3b	UN(2,2)	schoolID	13.1962	3.6505	0.07102	.
Cov3b	Residual		236.78	2.9456	0.00004	.
Cov4a	UN(1,1)	schoolID	13.9061	2.6583	-0.00823	-0.010080
Cov4a	UN(2,2)	schoolID	12.2230	3.4686	0.13953	0.068510
Cov4a	Residual		235.73	2.9325	0.00450	0.004452

Model 4b: Add Random Slope for Cluster-Mean-Centered Paid/Reduced vs Free Lunch

Level 1: $Math_{pc} = \beta_{0c} + \beta_{1c}(PaidvRedFree_{pc}) + \beta_{2c}(PaidvRedFree_{pc} - CMPaidvRedFree_{c}) + \beta_{3c}(PaidRedvFree_{pc}) + \beta_{4c}(PaidRedvFree_{pc} - CMPaidRedvFree_{c}) + e_{pc}$

Level 2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMPaidvRedFree_{c} - .30) + \gamma_{02}(CMPaidvRedFree_{c} - .30)^2 + \gamma_{03}(CMPaidRedvFree_{c} - .19) + U_{0c}$

$\beta_{1c} = \gamma_{10} + \gamma_{11}(CMPaidvRedFree_{c} - .30); \quad \beta_{2c} = U_{2c}$

$\beta_{3c} = \gamma_{30}; \quad \beta_{4c} = U_{4c}$

```
display "STATA Model 4b: Add Random Slope for Cluster-MC Paid/Reduced vs Free Lunch"
mixed math c.PvRF c.CM_PvRF30 c.PvRF#c.CM_PvRF30 c.CM_PvRF30#c.CM_PvRF30 ///
           c.PRvF c.CM_PRvF19, || schoolID: WC_PvRF WC_PRvF, ///
           cov(un) reml dfmethod(satterthwaite) dftable(pvalue) nolog
display "-2LL = " e(ll)*-2        // Print -2LL for model
estimates store Rand2             // Save fit for LRT
*lrtest Rand2 Fix2                // LRT for second random slope --> broken here!
```

STATA says ????

```
------------------------------------------------------------------------------
  Random-effects Parameters  |   Estimate   Std. Err.     [95% Conf. Interval]
-----------------------------+------------------------------------------------
schoolID: Unstructured       |
                var(WC_PvRF) |   8.976288          .             .           .
                var(WC_PRvF) |    .716911          .             .           .
                  var(_cons) |   13.95771          .             .           .
        cov(WC_PvRF,WC_PRvF) |   2.056415          .             .           .
          cov(WC_PvRF,_cons) |  -4.477288          .             .           .
          cov(WC_PRvF,_cons) |   -2.72332          .             .           .
-----------------------------+------------------------------------------------
               var(Residual) |   235.6973          .             .           .
------------------------------------------------------------------------------
```

```
print("R Model 4b: Add Random Slope for Cluster-MC Paid/Reduced vs Free Lunch")
print("R re-ordered new main effects and quadratic before interaction")
Model4b = lmer(data=Example4, REML=TRUE, formula=math~1+PvRF+CM_PvRF30+PRvF+CM_PRvF19
               +I(CM_PvRF30^2) +PvRF:CM_PvRF30 +(1+WC_PvRF+WC_PRvF|schoolID))
print("Show results using Satterthwaite DDF including -2LL as deviance")
llikAIC(Model4b, chkREML=FALSE); summary(Model4b, ddf="Satterthwaite")
```

```
       AIC        BIC     logLik   deviance   df.resid
108817.085 108921.791 -54394.543 108789.085  13068.000

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 schoolID (Intercept)  13.9568  3.73588
          WC_PvRF       8.9749  2.99581 -0.400
          WC_PRvF       0.7173  0.84694 -0.861  0.811
 Residual             235.6974 15.35244

Fixed effects:
                Estimate Std. Error        df t value  Pr(>|t|)
 (Intercept)     50.67170    0.66008  97.31881 76.7658 < 2.2e-16
PvRF            -5.90915    0.67065 137.01654 -8.8111 4.917e-15
CM_PvRF30      -21.79979    3.95430  98.45133 -5.5129 2.839e-07
PRvF            -4.33674    0.58185 461.05032 -7.4533 4.537e-13
CM_PRvF19        5.57247    4.59865  87.87255  1.2118   0.22885
I(CM_PvRF30^2) -11.27067    9.55994  93.98947 -1.1789   0.24139
PvRF:CM_PvRF30   5.70173    2.79450  93.86953  2.0403   0.04413
```

```
optimizer (nloptwrap) convergence code: 0 (OK)
boundary (singular) fit: see help('isSingular')
```

This warning indicates some estimation problems (although it's not obvious from the lmer results).

```
print("LRT for second random slope"); ranova(Model4b)
                                                   npar   logLik    AIC      LRT Df   Pr(>Chisq)
<none>                                               14 -54394.5 108817
WC_PvRF in (1 + WC_PvRF + WC_PRvF | schoolID)        11 -54408.7 108839 28.28033  3 0.0000031718
WC_PRvF in (1 + WC_PvRF + WC_PRvF | schoolID)        11 -54395.4 108813  1.79253  3      0.61656
```

Estimated G Correlation Matrix					
Row	Effect	School ID number	Col1	Col2	Col3
1	Intercept	125	1.0000	-0.3395	-1.0000
2	WC_PvRF	125	-0.3395	1.0000	1.0000
3	WC_PRvF	125	**-1.0000**	**1.0000**	1.0000

Covariance Parameter Estimates					
Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	schoolID	14.1532	2.7364	5.17	<.0001
UN(2,1)	schoolID	-3.5137	3.0392	-1.16	0.2476
UN(2,2)	schoolID	7.5665	3.8277	1.98	0.0240
UN(3,1)	schoolID	-4.5183	3.3103	-1.36	0.1723
UN(3,2)	schoolID	3.4114	2.2131	1.54	0.1232
UN(3,3)	schoolID	**8.18E-16**	.	.	.
Residual		235.76	2.9333	80.37	<.0001

SAS Says:

The second random slope variance was estimated as 0, which then was not counted in the DF below. The correlations with the 0-variance slope also went to their boundaries.

A 0 random slope variance or the non-significant LRT indicates we do not need it in the model. But we could still see if the PRvF slope wants to be systematically varying… one more model!

Likelihood Ratio Test for Fit4a vs. FitRand2

Name	Neg2LogLike	Parms	AIC	BIC	DevDiff	DFdiff	Pvalue
Fit4a	108791	4	108799	108809	.	.	.
FitRand2	108788	6	108800	108815	2.70163	2	0.25903

Btw, slope reliability = .056 using WC_PRvF variance = .140

Model 5a: Add 2 Interaction Slopes for Paid/Reduced vs Free Lunch—The Last Model!

Level 1: $Math_{pc} = \beta_{0c} + \beta_{1c}(PaidvRedFree_{pc})$
$+ \beta_{2c}(PaidvRedFree_{pc} - CMPaidvRedFree_c)$
$+ \beta_{3c}(PaidRedvFree_{pc})$
$+ \beta_{4c}(PaidRedvFree_{pc} - CMPaidRedvFree_c) + e_{pc}$

Level 2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMPaidvRedFree_c - .30) + \gamma_{02}(CMPaidvRedFree_c - .30)^2$
$+ \gamma_{03}(CMPaidRedvFree_c - .19) + \gamma_{04}(CMPaidRedvFree_c - .19)^2 + U_{0c}$
$\beta_{1c} = \gamma_{10} + \gamma_{11}(CMPaidvRedFree_c - .30)$; $\beta_{2c} = U_{2c}$
$\beta_{3c} = \gamma_{30} + \gamma_{31}(CMPaidRedvFree_c - .19)$

Simple L2 PvRF Between Slope $= \gamma_{10} + \gamma_{01}$
L2 PvRF Between*CM_PvRF30 $= \gamma_{11} + \gamma_{02}$
Simple L2 PRvF Between Slope $= \gamma_{30} + \gamma_{03}$
L2 PRvF Between*CM_PRvF19 $= \gamma_{31} + \gamma_{04}$

```
display "STATA Model 5a: Add 2 Interaction Slopes for Paid/Reduced vs Free Lunch"
mixed math c.PvRF c.CM_PvRF30 c.PvRF#c.CM_PvRF30 c.CM_PvRF30#c.CM_PvRF30  ///
           c.PRvF c.CM_PRvF19 c.PRvF#c.CM_PRvF19 c.CM_PRvF19#c.CM_PRvF19, ///
           || schoolID: WC_PvRF, /// Only PvRF random slope, PRvF is systematically varying
           cov(un) reml dfmethod(satterthwaite) dftable(pvalue) nolog
display "-2LL = " e(ll)*-2        // Print -2LL for model
lincom c.PvRF*1 + c.CM_PvRF30*1                          // Simple PvRF L2 Between
lincom c.PvRF#c.CM_PvRF30*1 + c.CM_PvRF30#c.CM_PvRF30*1  // L2 PvRF Between*CM_PvRF30
lincom c.PRvF*1 + c.CM_PRvF19*1                          // Simple L2 PRvF Between
lincom c.PRvF#c.CM_PRvF19*1 + c.CM_PRvF19#c.CM_PRvF19*1  // L2 PRvF Between*CM_PRvF19
predict pred5a
corr math pred5a                 // Get total r to make R2
display "Total-R2 = " r(rho)^2  // Print total R2 relative to empty model
```

```
print("R Model 5a: Add 2 Interaction Slopes for Paid/Reduced vs Free Lunch")
print("R re-ordered all main effects and quadratics before interactions")
Model5a = lmer(data=Example4, REML=TRUE, formula=math~1+PvRF+CM_PvRF30+PRvF+CM_PRvF19
               +I(CM_PvRF30^2) +I(CM_PRvF19^2) +PvRF:CM_PvRF30 +PRvF:CM_PRvF19
               +(1+WC_PvRF|schoolID))
print("Show results using Satterthwaite DDF including -2LL as deviance")
llikAIC(Model5a, chkREML=FALSE); summary(Model5a, ddf="Satterthwaite")
```

```
      AIC        BIC     logLik   deviance   df.resid
108802.489 108899.716 -54388.244 108776.489  13069.000

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 schoolID (Intercept)  14.005   3.7423
          WC_PvRF      12.158   3.4868   -0.484
 Residual             235.737  15.3537
```

Fixed effects from Model 4a:

	Estimate	Std. Error
(Intercept)	**50.69385**	0.66263
PvRF	**-5.83385**	0.70117
CM_PvRF30	**-22.39799**	3.98457
PRvF	**-4.46526**	0.56777
CM_PRvF19	**6.47672**	4.57263
I(CM_PvRF30^2)	**-11.81434**	9.64630
PvRF:CM_PvRF30	**5.67466**	2.82137

Fixed effects:

	Estimate	Std. Error	df	t value	Pr(>|t|)
(Intercept)	**50.78322**	0.67570	98.35997	75.1568	< 2.2e-16
PvRF	**-5.99698**	0.73340	250.45675	-8.1770	1.462e-14
CM_PvRF30	**-17.91717**	6.60158	95.31084	-2.7141	0.007889
PRvF	**-4.13153**	0.71235	12737.82690	-5.7999	6.794e-09
CM_PRvF19	**-2.19192**	10.76289	87.98372	-0.2037	0.839093
I(CM_PvRF30^2)	**-31.39326**	22.15125	94.79414	-1.4172	0.159696
I(CM_PRvF19^2)	**32.33294**	32.20325	86.92435	1.0040	0.318152
PvRF:CM_PvRF30	**7.00902**	3.29879	186.43784	2.1247	0.034927
PRvF:CM_PRvF19	**-2.51268**	3.30184	6507.73730	-0.7610	0.446689

Which fixed effects have changed in their interpretation? But should we trust this model???

```
print("Simple L2 PvRF Between");    contest1D(Model5a, L=c(0,1,1,0,0,0,0,0,0))
print("L2 PvRF Between*CM_PvRF30"); contest1D(Model5a, L=c(0,0,0,0,0,1,0,1,0))
print("Simple PRvF L2 Between");    contest1D(Model5a, L=c(0,0,0,1,1,0,0,0,0))
print("L2 PRvF Between*CM_PRvF19"); contest1D(Model5a, L=c(0,0,0,0,0,0,1,0,1))
```

	Estimate	**Std. Error**	**df**	**t value**	**Pr(>|t|)**
1	-23.914155	6.5947867	95.083392	-3.6262211	0.00046467679
1	-24.384232	21.716517	87.902905	-1.1228427	0.26456179
1	-6.3234446	10.748645	87.51161	-0.58830155	0.55784519
1	29.820257	31.913361	83.788256	0.93441292	0.35277607

```
print("Psuedo-R2 relative to single random slope model using Jonathan's function")
pseudoRSquaredinator(smallerModel=Model2c, largerModel=Model5a)
```

R2 Random.(Intercept)	R2 Random.WC_PvRF	R2 L1.sigma2
-0.0148076903	0.1443460500	0.0044557402

```
print("Total-R2 relative to empty means model using Jonathan's function")
totalRSquaredinator(data=Example4, dvName="math", model=Model5a)
0.16899655
```

Pseudo-R2 Relative to CovRand (from SAS)
Change in Pseudo-R2 for Cov3b vs. Cov5a

Name	CovParm	Subject	Estimate	StdErr	PseudoR2	PseudoR2Change
CovRand	UN(1,1)	schoolID	13.7927	2.6170	.	.
CovRand	UN(2,2)	schoolID	14.2050	3.8075	.	.
CovRand	Residual		236.79	2.9458	.	.
Cov3b	UN(1,1)	schoolID	13.7671	2.6236	0.00185	.
Cov3b	UN(2,2)	schoolID	13.1962	3.6505	0.07102	.
Cov3b	Residual		236.78	2.9456	0.00004	.
Cov5a	UN(1,1)	schoolID	13.9797	2.6843	-0.01356	-0.015416
Cov5a	UN(2,2)	schoolID	12.1583	3.4590	0.14409	0.073063
Cov5a	Residual		235.74	2.9327	0.00445	0.004405

Although the level-1 residual variance was slightly reduced from the PRvF*CM_PRvF19 interaction, the level-2 random intercept variance increased (perhaps as a result). Meanwhile, the level-2 random slope variance decreased even though no new fixed effects were targeting it specifically.

This is why we call them “pseudo-R^2”!

Sample Results Section starts here, focusing on within and contextual effects

[indicates notes about what to customize or also include; note that SE and p-values are not needed if you provide tables for the model solutions]

Note that the smushed results are not reported, and results are combined across models to give all fixed slopes of interest (so not all models are reported)…

The extent to which student math outcomes could be predicted from student-level (and corresponding school-level) variables of ordinal lunch status (fully paid, reduced-price, or free) was examined in a series of multilevel models in which the 13,802 students were modeled as nested within their 94 schools. Restricted Maximum likelihood (REML) within SAS MIXED [or STATA MIXED or R `lmer`] was used in estimating and reporting all model parameters. The significance of fixed effects was evaluated with univariate Wald tests using Satterthwaite denominator degrees of freedom. Alpha was chosen as .05. Model-implied fixed effects were requested via ESTIMATE [or LINCOM or `contest1D`] statements. Effect size for the fixed effects was evaluated via pseduo-R^2 values for the proportion reduction in each variance component relative to a nested model without the predictors in question, as well as with total-R^2, the squared correlation between the actual math outcomes and those predicted by the model fixed effects.

As derived from an empty means, random intercept model, student math had an intraclass correlation of ICC = .152, indicating that 15.2% of the variance in student math was due to mean differences between schools, a significant amount, $-2\Delta LL(1) = 1860.20$, $p < .0001$. Given an average of 139 students per school in this sample, the ICC = .152 translated into a design effect = 22.03 (and effective sample size ≈ 594), further indicating the need for a multilevel analysis. The school mean math outcomes had strong reliability, as evidenced by an ICC2 = .961. The fixed intercept was 47.756 (SE = 0.723), which represented the expected average school mean math outcome. A random intercept confidence interval (computed as the fixed intercept ± 1.96*SQRT[random intercept variance]) indicated that 95% of the schools were expected to have school mean math outcomes between 34.542 and 60.970 (around the average of 47.756).

For the ordinal student lunch variable, an ordinal version of the two-level model (i.e., with a cumulative logit link function and a multinomial level-1 conditional distribution) was estimated instead. Using $\pi^2 / 3$ for the model-scale residual variance, the ICC = .351, which was also significantly greater than 0, $-2\Delta LL(1) = 2{,}981.76$, $p < .0001$. Consequently, the effects of student lunch were modeled at both levels. For level-1 students, the ordinal lunch variable was represented via two sequentially-coded predictors: for paid=0 versus reduced or free=1, and paid or reduced=0 versus free=1. School means were then computed for these predictors and centered at their sample averages (proportions of .30 and .19, respectively).

We first examined the effects of level-1 student and level-2 school paid versus reduced or free lunch, which accounted for 5.4% of the level-1 residual variance in math and 69.4% of the level-2 random intercept variance in math. The fixed intercept was 50.603 (SE = 0.439), which represented the expected math outcome for a student with paid lunch from a school where 70% of the students had paid lunch. At level 1, the within-school lunch slope was significantly negative (Est = −9.173, SE = 0.334, $p < .0001$), indicating that student math was expected to be significantly lower by 9.173 for students who received reduced or free lunch (as compared to paid lunch). At level 2, the contextual-school lunch slope was also significantly negative (Est = –16.843, SE = 2.025, $p < .0001$), indicating that after controlling for student lunch status, school mean math was expected to be significantly lower by 1.684 for every 10% more students with reduced or free lunch in that school.

We then examined the extent of school differences in the within-school lunch-related disadvantage by adding a level-2 random slope (and its covariance with the level-2 random intercept); we used a cluster-mean-centered version of the level-1 lunch predictor to prevent random conflation. Model fit improved significantly, −2ΔLL(2) = 89.27, $p < .0001$, indicating significant between-school heterogeneity in the within-school lunch-related disadvantage. The level-1 within-school fixed slope was then −8.439 (SE = 0.584), which represented the average lunch-related disadvantage in student math across schools. A random slope confidence interval (computed as the fixed slope ± 1.96*SQRT[random slope variance]) indicated that 95% of the schools were expected to have a lunch-related disadvantage between 1.050 and 15.827 (around the average of −8.439). However, slope reliability was only .637, likely limited by the binary measurement of the student lunch predictor.

We then examined moderation of the within-school lunch disadvantage by school lunch composition by allowing the within-school and contextual lunch slopes to interact with the school mean lunch predictor, which explained 7.10% of the new random lunch slope variance and an additional 0.2% of the random intercept variance across schools (i.e., that remained in the prior random slope model). Although neither interaction was significant, they were retained in subsequent models examining the additional difference between students with reduced versus free lunch. Following the same process, we examined effects of level-1 student and level-2 school reduced versus free lunch, which accounted for another 0.4% of the level-1 residual variance in math, 0% of the level-2 random intercept variance in math, and (unexpectedly) 6.9% of the level-2 random slope variance (for the disavantage of reduced or free lunch relative to paid lunch). A level-2 random slope for between-school differences in the within-school effect of reduced versus free lunch did not significantly improve model fit and resulted in convergence problems, and thus it was removed. Finally, we examined moderation of the reduced versus free lunch effects by allowing their within-school and contextual slopes to interact with their school mean predictor, which resulted in model instability (i.e., highly inflated standard errors). Thus, the model with only main effects for reduced versus free lunch was retained. Results for the model below are shown in Table 1 and can be interpreted as follows.

Level 1:
$$
\begin{aligned}
Math_{pc} = \beta_{0c} &+ \beta_{1c}\left(PaidvRedFree_{pc}\right) \\
&+ \beta_{2c}\left(PaidvRedFree_{pc} - CMPaidvRedFree_{c}\right) \\
&+ \beta_{3c}\left(PaidRedvFree_{pc}\right) + e_{pc}
\end{aligned}
$$

Level 2:
$$
\begin{aligned}
\beta_{0c} &= \gamma_{00} + \gamma_{01}(CMPaidvRedFree_{c} - .30) + \gamma_{02}(CMPaidvRedFree_{c} - .30)^{2} \\
&\quad + \gamma_{03}(CMPaidRedvFree_{c} - .19) + U_{0c} \\
\beta_{1c} &= \gamma_{10} + \gamma_{11}(CMPaidvRedFree_{c} - .30); \quad \beta_{2c} = U_{2c} \\
\beta_{3c} &= \gamma_{30}
\end{aligned}
$$

Table 1

	Fixed effects	Estimate	SE	$p <$
γ_{00}	Intercept	50.694	0.663	.001
	PvRF: Paid=0 versus Reduced or Free Lunch=1			
γ_{10}	Within-School Simple Main Effect	−5.834	0.701	.001
γ_{01}	Linear Contextual Main Effect	−22.398	3.985	.001
γ_{02}	Quadratic Contextual Main Effect	−11.814	9.646	.224
γ_{11}	Within-School by Linear Contextual Interaction	5.675	2.821	.047
	PRvF: Paid=0 versus Reduced or Free Lunch=1			
γ_{30}	Within-School Main Effect	−4.465	0.568	.001
γ_{03}	Linear Contextual Main Effect	6.477	4.573	.160

	Model for the Variance	Estimate
$\tau^2_{U_0}$	Level-2 School Random Intercept Variance	13.934
$\tau^2_{U_2}$	Level-2 School Random Within-School PvsRF Slope Variance	12.221
	Level-2 Random Effects Correlation	−0.477
σ^2_e	Level-1 Residual Variance	235.726

The intercept (50.694) is the expected math outcome for a student who pays for lunch and who attends a school where 70% of students pay for lunch, 9% receive reduced-price lunch, and 22% receive free lunch. Let us first consider differences between students with reduced-price for lunch relative to paid lunch. With respect to the level-1 within-school slope (−5.834), relative to their peers with paid lunch at the same school, students with reduced-price lunch were predicted to have significantly lower math by 5.834 (specifically for schools in which 30% of students received reduced or free lunch, given its cross-level interaction with the school mean predictor). With respect to the level-2 contextual slope (−22.398), school mean math was significantly lower by 2.240 per 10% more children who received reduced rather than paid lunch (specifically for students with paid lunch given the cross-level interaction, and specifically for schools in which 30% of students received reduced or free lunch given the level-2 interaction). The cross-level interaction (5.675) indicated that the within-school disadvantage for reduced relative to paid lunch was significantly less negative (weaker) by 0.568 for every 10% more students who received reduced or free lunch in that school. The level-2 interaction (−11.814) indicated that the contextual-school lunch effect became nonsignificantly more negative (stronger) by 1.181 for every 10% more students who received reduced or free lunch in that school. Finally, the two remaining slopes further distinguish students with free lunch from those with reduced lunch. With respect to the level-1 within-school slope (−4.465), relative to their peers with reduced-price lunch at the same school, students with free lunch were predicted to have significantly lower math by 4.465 (which was unconditional with respect to school composition given its lack of inclusion in any interaction terms). With respect to the level-2 contextual slope (6.477), school mean math was nonsignificantly higher by 0.648 per 10% more children who received free rather than reduced lunch.

These results are also summarized in Figures 1–3. [Note: predicted values are shown for pedagogical purposes but would not normally be included in a figure.] First, as shown by the x-axis in both figures, students who paid full-price for lunch had the highest predicted math outcomes, followed by students who paid reduced-price lunch, who in turn had higher predicted math outcomes than students who received free lunch. In addition, as shown by the growing vertical distance between the lines in Figure 1, schools with a greater proportion of students receiving reduced-price lunch had lower average math outcomes, and this effect became nonsignificantly stronger as the school proportion increased. In addition, as shown in Figure 1 by the difference in slope from paid to reduced-price lunch on the x-axis, that student disadvantage was significantly smaller in schools with more children receiving reduced-price lunch (holding constant the proportion of students receiving free lunch). However, there was no model-predicted difference in slope from reduced-price to free lunch by school composition (i.e., no moderation). Figure 2 shows the same pattern of growing negative contextual effects for schools with a higher proportion of students receiving free lunch (rather than paid lunch, holding constant the proportion receiving reduced lunch). However, as shown by the lack of vertical distance between the lines in Figure 3, a nonsignificant tendency in the opposite direction was found for the proportion of students receiving free lunch instead of reduced lunch—a positive and constant contextual effect instead (holding constant the proportion of students receiving paid lunch). Thus, at the school level, the proportion of students with paid lunch is the most salient distinction in predicting math outcomes.

Figure 1: Contextual Effect of School Proportion of Students Receving Reduced-Price Lunch (Rather than Paid)

Holding Constant School Proportion Free Lunch:			
School Composition:	**Paid**	**Reduced**	**Free**
10% Reduced, 10% Free	52.233	45.831	41.366
20% Reduced, 10% Free	50.111	44.277	39.812
30% Reduced, 10% Free	47.753	42.487	38.021
40% Reduced, 10% Free	45.159	40.460	35.995
Disadvantages:		**PvR:**	**RvF:**
10% Reduced, 10% Free		-6.401	-4.465
20% Reduced, 10% Free		-5.834	4.465
30% Reduced, 10% Free		-5.266	4.465
40% Reduced, 10% Free		-4.699	4.465

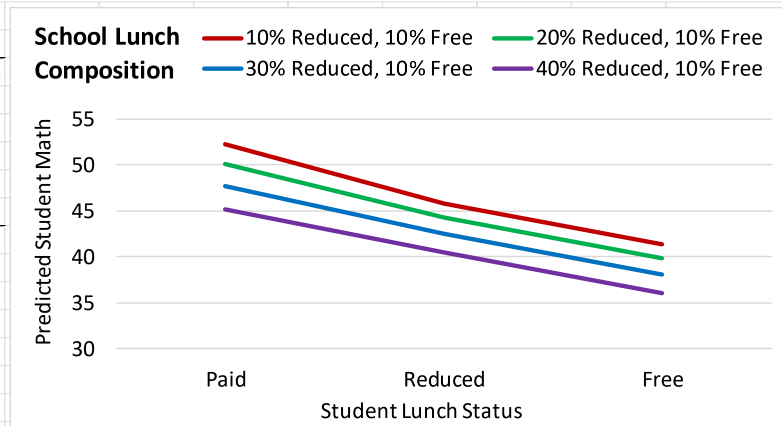

Figure 3: Contextual Effect of School Proportion of Students Receving Free Lunch (Rather than Paid)

Holding Constant School Proportion Reduced Lunch:			
School Composition:	**Paid**	**Reduced**	**Free**
10% Reduced, 10% Free	52.233	45.831	41.366
10% Reduced, 20% Free	50.759	44.925	40.460
10% Reduced, 30% Free	49.048	43.782	39.317
10% Reduced, 40% Free	47.102	42.403	37.938
Disadvantages:		**PvR:**	**RvF:**
5% Reduced, 10% Free		-6.401	-4.465
10% Reduced, 20% Free		-5.834	4.465
15% Reduced, 30% Free		-5.266	4.465
20% Reduced, 40% Free		-4.699	4.465

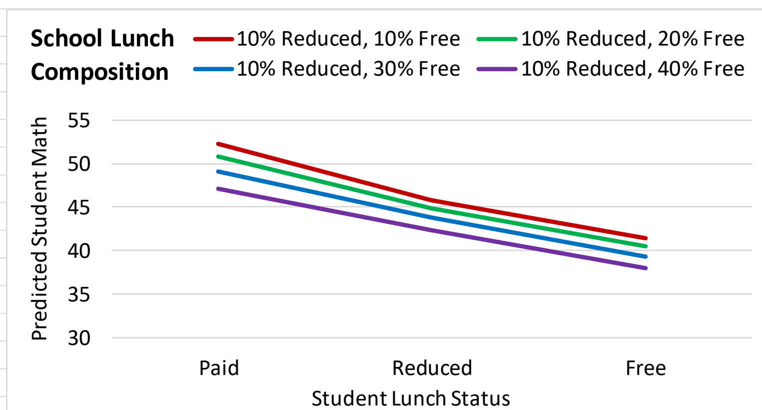

Figure 3: Contextual Effect of School Proportion of Students Receving Free Lunch (Rather than Reduced)

Holding Constant School Proportion Paid Lunch:			
School Composition:	**Paid**	**Reduced**	**Free**
30% Reduced, 10% Free	47.753	42.487	38.021
20% Reduced, 20% Free	48.401	43.134	38.669
10% Reduced, 30% Free	49.048	43.782	39.317
0% Reduced, 40% Free	49.696	44.430	39.964
Disadvantages:		**PvR:**	**RvF:**
30% Reduced, 10% Free		-5.266	-4.465
20% Reduced, 20% Free		-5.266	4.465
10% Reduced, 30% Free		-5.266	4.465
0% Reduced, 40% Free		-5.266	4.465