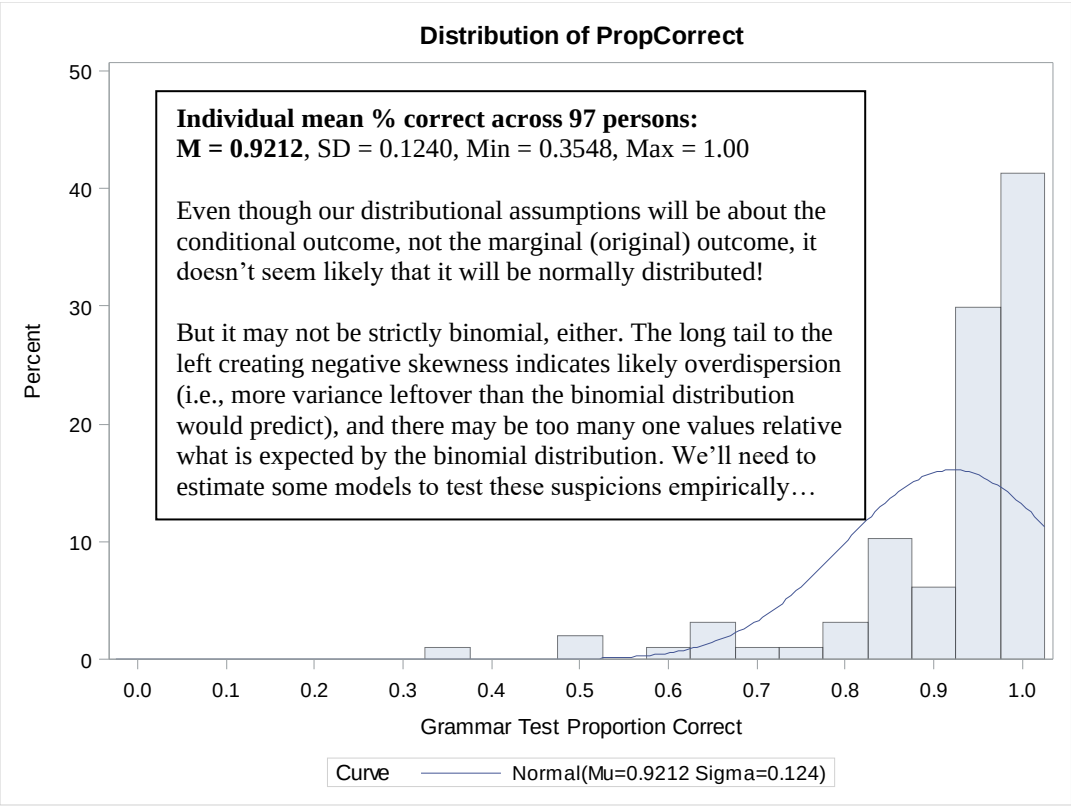
Example 4a: Generalized Linear Models for Outcomes with Ceiling and Floor Effects
(complete syntax and output available for STATA, R, and SAS electronically)

The real data for this example come from the publication below, which examined annual growth in a test of grammatical understanding from Kindergarten through 4th grade in children with non-specific language impairment (NLI) or specific language impairment (SLI):

Rice, M. L., Tomblin, J. B., **Hoffman, L.**, Richman, W. A., & Marquis, J. (2004). Grammatical tense deficits in children with SLI and nonspecific language impairment: Relationships with nonverbal IQ over time. *Journal of Speech-Language-Hearing Research, 47*(4), 816–834.

The current example is a cross-sectional analysis of how grammatical understanding at third grade (measured by proportion correct on a test of grammatical understanding) is predicted by group (NLI=0, SLI=1) and mother's years of education (centered so that 0=12 years). Our outcome is **proportion correct**, which is bounded by 0 and 1, and so we will use a logit link and the binomial family of conditional response distributions. **Because the binomial is a discrete distribution, we will need to parameterize the model to predict the number of correct responses out of the number of trials directly instead of proportion correct.** This example will also demonstrate two ways of addressing binomial overdispersion: *additive* (through individual random intercepts) and *multiplicative* (through the beta-binomial distribution), as well as zero-inflated (actually one-inflated here; stay tuned) versions of the binomial and beta-binomial model variants (in which the probability of being an extra zero is predicted in a separate submodel using a logit link).

In SAS (as given in online materials only), I am still using GLIMMIX for the binomial models, as well as FMM (finite mixture model) for the beta-binomial and zero-inflated model variants. In STATA, I am using GLM to get conditional distribution fit, as well as MEGLM, BETABIN, ZIB, and ZIBBIN. In R, I am using the VGLM function from the VGAM package and GLMER from the LME4 package. Unfortunately, because the VGLM function uses expected information instead of observed information (as used in STATA and SAS), the standard errors for the parameter estimates (and thus any Wald test results) will differ between STATA/SAS and R. Likelihood ratio test results are the same, however. Btw, in SAS GLIMMIX, I set denominator DF to "none" so that the SAS Wald test results will match those of STATA.



STATA Syntax for Importing and Preparing Data for Analysis:

```
// Defining global variable for file location to be replaced in code below
// \\Client\ precedes path in Virtual Desktop outside H drive;
global filesave "C:\Dropbox\24_PSQF6270\PSQF6270_Example4a"

// Import Example 4a Stata data
use "$filesave\PSQF6270_Example4a.dta", clear

// Label existing variables for analysis
label variable nlivsli     "Group: 0=NLI, 1=SLI"
label variable momed12     "Mother Education (0=12 years)"
label variable propcorrect "Grammar Test Proportion Correct"
// Create number correct for denominator of binomial outcome
gen ntrials=100
gen ncorrect=round(propcorrect*ntrials,1)
// Compute number incorrect for zero-inflated binomial model
gen nincorrect=ntrials-ncorrect
gen propincorrect=1-propcorrect

// Find betabin and zbin, install before continuing
// search betabin
// search zbin

display "STATA Distribution of Proportion Correct"
summarize propcorrect
hist propcorrect, percent start(0) width(.05)
graph export "$filesave\STATA Proportion Correct Histogram.png", replace
```

R Syntax for Importing and Preparing Data for Analysis (after loading packages *haven*, *TeachingDemos*, *psych*, *multcomp*, *vgam*, and *lme4* as shown online):

```
# Define variables for working directory and data name
filesave = "C:\\Dropbox/24_PSQF6270/PSQF6270_Example4a/"
filename = "PSQF6270_Example4a.sas7bdat"
setwd(dir=filesave)

# Import Example 4a SAS data
Example4a = read_sas(data_file=paste0(filesave,filename))
# Convert to data frame without labels to use for analysis
Example4a = as.data.frame(Example4a)

# Label existing variables for analysis
#NLIvSLI=     "Group: 0=NLI, 1=SLI"
#momed12=     "Mother Education (0=12 years")
#PropCorrect= "Grammar Test Proportion Correct"

# Create number correct for denominator of binomial outcome
Example4a$Ntrials=100
Example4a$Ncorrect=Example4a$PropCorrect*Example4a$Ntrials
Example4a$Ncorrect=round(Example4a$Ncorrect,0)
# Compute number incorrect for zero-inflated binomial model
Example4a$Nincorrect=Example4a$Ntrials-Example4a$Ncorrect
Example4a$PropIncorrect=1-Example4a$PropCorrect

print("R Distribution of Proportion Correct")
describe(x=Example4a$PropCorrect)

# To save a plot: open a file, create the plot, then close the file
png(file = "R Proportion Correct Histogram.png")  # open file
hist(x=Example4a$PropCorrect, freq=FALSE,
     ylab="Density",xlab="Grammar Test Proportion Correct") # axis labels
dev.off()  # close file
```

1) Empty Binomial Model for % correct
DV = Events/Trials in SAS and STATA; Events/Non-Events in R

$\#Correct_i \sim Binomial(p_i, Ntrials_i)$ → **p_i is probability of any one trial being correct as the "event"**
$Logit(p_i$ for correct trial$) = \beta_0$
Conditional mean μ for $\#Correct_i = Ntrials_i * p_i$
Conditional variance for $\#Correct_i = (Ntrials_i * p_i)(1 - p_i)$

STATA Syntax and Partial Output for Empty Binomial Model:

```
display "STATA Empty Binomial Model using glm -- ntrials is denominator"
glm ncorrect, link(logit) family(binomial ntrials) nolog

Generalized linear models                         No. of obs      =         97
Optimization     : ML                             Residual df     =         96
                                                  Scale parameter =          1
Deviance         =   1620.05009                   (1/df) Deviance =   16.87552
Pearson          =  2041.435988                   (1/df) Pearson  =   21.26496 → way too high! (1=good)
Variance function: V(u) = u*(1-u/ntrials)         [Binomial]
Link function    : g(u) = ln(u/(ntrials-u))       [Logit]
                                                  AIC             =   19.00196 → not usual version!
Log likelihood   = -920.5951086                   BIC             =   1180.878 → not usual version!
------------------------------------------------------------------------------
             |                 OIM
    ncorrect |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _cons |   2.459276   .0376936    65.24   0.000     2.385397    2.533154  mean in logits
------------------------------------------------------------------------------  → probability = .9212
```

To inverse link from logits to predicted % correct:

$$Prob(y = 1) = \frac{\exp(2.4593)}{1 + \exp(2.4593)} = .9212$$

The sample mean probability of getting any item correct is .9212. So even though we are actually modeling number of correct trials as the outcome, the model directly predicts the logit of the **proportion correct** (as the conditional mean *p*, the probability that any trial = 1).

```
display "-2LL= " e(ll)*-2  // Print -2LL for model
-2LL= 1841.1902

margins   // Get intercept in expected # events (ILINK*Ntrials = percent here)
------------------------------------------------------------------------------
             |            Delta-method
             |     Margin   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _cons |   92.12371   .2735021   336.83   0.000     91.58766    92.65977 mean in # events
------------------------------------------------------------------------------
```

R Syntax and Partial Output for Empty Binomial Model:

```
# Save sample size as variable for use in Pearson chi-square/DF fit
N=97

print("R Empty Binomial Model using vglm and two outcome columns")
ModelEmpty = vglm(data=Example4a, binomialff(link="logitlink", multiple.responses=FALSE),
                  formula=cbind(Ncorrect,Nincorrect)~1) # Can also use multiv format of 0/1
print("Print ML -2LL with results"); -2*logLik(ModelEmpty); summary(ModelEmpty)
[1] 1841.1902 → -2LL for model

Coefficients:
            Estimate Std. Error z value             Pr(>|z|)
 (Intercept) 2.459276   0.037693  65.246 < 0.00000000000000022 mean in logits → probability = .9212

Name of linear predictor: logitlink(prob)
Residual deviance: 1620.0501 on 96 degrees of freedom
Log-likelihood: -920.59511 on 96 degrees of freedom
```

```
Warning: Hauck-Donner effect detected in the following estimate(s):
'(Intercept)'
```

This warning indicates that the intercept had some difficulty being estimated. For more info, see https://search.r-project.org/CRAN/refmans/VGAM/html/hdeff.html

```
print("Get intercept in probability")
ModelEmptyProb=1/(1+exp(-1*coefficients(ModelEmpty))); ModelEmptyProb
0.92123711

print("Pearson Chi-Square / DF Index of Fit")
sum(residuals(ModelEmpty, type="pearson")^2)/(N-1)  # STATA N-k
[1] 21.262847 → way too high (1=good)
```

---

2) Two-Predictor Binomial Model

$\#Correct_i \sim Binomial(p_i, Ntrials_i)$ → **p_i is probability of any one trial being correct as the "event"**

$Logit(p_i \text{ for correct trial}) = \beta_0 + \beta_1(NLIvSLI_i) + \beta_2(MotherEd_i - 12)$

Conditional mean μ: $\#Correct_i = Ntrials_i * p_i$

Conditional variance of $\#Correct_i = (Ntrials_i * p_i)(1 - p_i)$

STATA Syntax and Partial Output for Two-Predictor Binomial Model:

```
display "STATA Two-Predictor Binomial Model using glm"
glm ncorrect c.nlivsli c.momed12, link(logit) family(binomial ntrials) nolog

Generalized linear models                         No. of obs      =         97
Optimization     : ML                             Residual df     =         94
                                                  Scale parameter =          1
Deviance         =  1310.593044                   (1/df) Deviance =   13.94248
Pearson          =  1448.891028                   (1/df) Pearson  =   15.41373 → still way too high!
Variance function: V(u) = u*(1-u/ntrials)         [Binomial]
Link function    : g(u) = ln(u/(ntrials-u))       [Logit]
                                                  AIC             =   15.85292
Log likelihood   = -765.8665858                   BIC             =   880.5702
------------------------------------------------------------------------------
             |                 OIM
    ncorrect |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     nlivsli |  -1.221578   .0858707   -14.23   0.000    -1.389881   -1.053275  Beta1
     momed12 |   .1193325   .0214268     5.57   0.000     .0773368    .1613283  Beta2
       _cons |   3.071929   .0746183    41.17   0.000      2.92568    3.218178  Beta0 logit → prob = .956
------------------------------------------------------------------------------

display "-2LL= " e(ll)*-2  // Print -2LL for model
-2LL= 1531.7332

estat ic, n(97)            // Print AIC and BIC
-----------------------------------------------------------------------------
       Model |        Obs  ll(null)  ll(model)      df         AIC        BIC
-------------+---------------------------------------------------------------
           . |         97         .  -765.8666       3    1537.733   1545.457
-----------------------------------------------------------------------------

test (c.nlivsli=0)(c.momed12=0)  // Multiv Wald test of model
          chi2(  2) =  273.58 → way too optimistic because distribution fit is bad
        Prob > chi2 =    0.0000

display "STATA Two-Predictor Binomial Model -- get odds ratios using eform"
glm ncorrect c.nlivsli c.momed12, link(logit) family(binomial ntrials) eform
------------------------------------------------------------------------------
             |                 OIM
    ncorrect | Odds Ratio   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     nlivsli |   .2947646   .0253116   -14.23   0.000     .2491048    .3487937 exp(Beta1)
     momed12 |   1.126745   .0241425     5.57   0.000     1.080406    1.175071 exp(Beta2)
       _cons |    21.5835   1.610525    41.17   0.000      18.6469    24.98257 exp(Beta0)
------------------------------------------------------------------------------
```

R Syntax and Partial Output for Two-Predictor Binomial Model:

```
print("R Two-Predictor Binomial Model using vglm")
ModelBin = vglm(data=Example4a, binomialff(link="logitlink", multiple.responses=FALSE),
                formula=cbind(Ncorrect,Nincorrect)~1+NLIvSLI+momed12)
print("Print -2LL, AIC, BIC, and results")
-2*logLik(ModelBin); AIC(ModelBin); BIC(ModelBin); summary(ModelBin)
[1] 1531.7332 → -2LL for model
[1] 1537.7332 → AIC
[1] 1545.4573 → BIC

Coefficients:
             Estimate Std. Error  z value             Pr(>|z|)
(Intercept)  3.071929   0.074610  41.1729 < 0.00000000000000022  Beta0 logit → prob = .956
NLIvSLI     -1.221578   0.085864 -14.2269 < 0.00000000000000022  Beta1
momed12      0.119333   0.021426   5.5695         0.00000002555  Beta2

Name of linear predictor: logitlink(prob)
Residual deviance: 1310.593 on 94 degrees of freedom
Log-likelihood: -765.86659 on 94 degrees of freedom

print("Pearson Chi-Square / DF Index of Fit")
sum(residuals(ModelBin, type="pearson")^2)/(N-3)  # STATA N-k
[1] 15.410798 → still way too high!

print("Multiv Wald test of model")
BinTestR2 = glht(model=ModelBin, linfct=c("NLIvSLI=0","momed12=0"))
print(summary(BinTestR2, test=Chisqtest()), digits="8") # Joint chi-square test
Global Test:
      Chisq DF    Pr(>Chisq)
1 273.62108  2 3.8365119e-60 → way too optimistic because distribution fit is bad

print("Get odds ratios with 95% CIs")
exp(cbind(OR=coefficients(ModelBin), confint.default(ModelBin)))
                    OR       2.5 %      97.5 %
(Intercept) 21.58349665 18.64718757 24.98217632 exp(Beta0)
NLIvSLI      0.29476465  0.24910818  0.34878902 exp(Beta1)
momed12      1.12674455  1.08040748  1.17506895 exp(Beta2)
```

Before interpreting these results, let's see if we can get better distribution fit. Here are some alternative models that incorporate overdispersion, zero-inflation (actually one-inflation here), or both at the same time…

---

3) Two-Predictor Binomial Model with Additive Overdispersion: Also known as adding an "observation-level random effect" (OLRE)

The random intercept variance is on the model-scale (in logits), and it effectively soaks up all discrepancy to each person's predicted logit.

$\#Correct_i \sim Binomial(p_i, Ntrials_i)$ → p_i is probability of any one trial being correct

$Logit(p_i \text{ for correct}) = \beta_0 + \beta_1(NLIvSLI_i) + \beta_2(MotherEd_i - 12) + U_{0i}$

Conditional mean μ of $\#Correct_i = Ntrials_i * p_i$

Conditional variance of $\#Correct_i = (Ntrials_i * p_i)(1 - p_i)$

STATA Syntax and Partial Output for Additive Overdispersion:

```
display "STATA Two-Predictor Binomial Model with Additive Overdispersion"
display "Using meglm instead; || id. adds random intercept variance"
display "Also known as observation-level random effect for overdispersion"
meglm ncorrect c.nlivsli c.momed12, || id: , ///
      link(logit) family(binomial ntrials) intmethod(laplace) nolog
// LRT for added random intercept variance is done for you automatically
// Multiv Wald test of model also given in meglm for you, but Pearson chi-square/DF not given

Mixed-effects GLM                               Number of obs     =         97
Family:                  binomial
Link:                       logit
Binomial variable:       ntrials
Group variable:               id                Number of groups  =         97
                                                Obs per group:
```

```
                                                      min =          1
Integration method:     laplace                       avg =        1.0
                                                      max =          1
                                              Wald chi2(2)      =      14.04 → Wald test of model
Log likelihood = -274.88176                   Prob > chi2       =     0.0009
------------------------------------------------------------------------------
    ncorrect |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     nlivsli |  -1.793682   .5089051    -3.52   0.000    -2.791118   -.7962467 Beta1 still signif
     momed12 |   .0327918   .1341283     0.24   0.807    -.2300949    .2956784 Beta2 no longer signif
       _cons |   4.742648   .4350784    10.90   0.000      3.88991    5.595386 Beta0 logit → prob=.991
-------------+----------------------------------------------------------------
id           |
   var(_cons)|   4.381075   1.028769                      2.765034    6.941619 extra variance in logits
------------------------------------------------------------------------------
LR test vs. logistic model: chibar2(01) = 981.97      Prob >= chibar2 = 0.0000 → LRT additive overdisp
```

Along with the much larger standard errors (as expected from allowing extra variance), the estimates have also changed because the total model has more variance in it now (as opposed to the total 3.29 residual variance given the logit link).

```
display "-2LL= " e(ll)*-2  // Print -2LL for model
-2LL= 549.76351

estat ic, n(97)            // Print AIC and BIC
-----------------------------------------------------------------------------
       Model |        Obs  ll(null)  ll(model)      df         AIC        BIC
-------------+---------------------------------------------------------------
           . |         97         .  -274.8818       4    557.7635   568.0624
-----------------------------------------------------------------------------

display "STATA Two-Predictor Binomial Model with Additive Overdispersion"
display "Using meglm instead and getting odds ratios using eform"
meglm ncorrect c.nlivsli c.momed12, || id: , ///
      link(logit) family(binomial ntrials) intmethod(laplace) eform nolog
------------------------------------------------------------------------------
    ncorrect |     exp(b)   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     nlivsli |   .1663465   .0846546    -3.52   0.000     .0613526    .4510186 exp(Beta1) → was 0.29
     momed12 |   1.033335   .1385995     0.24   0.807     .7944582    1.344038 exp(Beta2) → was 1.12
       _cons |   114.7376   49.91986    10.90   0.000     48.90649    269.1815 exp(Beta0) → was 21.58
-------------+----------------------------------------------------------------
```

R Syntax and Partial Output for Additive Overdispersion:

```
print("R Two-Predictor Binomial Model using glmer")
print("Additive Overdispersion via Random Intercept Variance (1|ID)")
print("Also known as observation-level random effect for overdispersion")
ModelBinAdd = glmer(data=Example4a, family=binomial(link="logit"),
                    formula=cbind(Ncorrect,Nincorrect)~1+NLIvSLI+momed12+(1|ID))
print("Print -2LL, AIC, BIC, and results")
-2*logLik(ModelBinAdd); AIC(ModelBinAdd); BIC(ModelBinAdd); summary(ModelBinAdd)
'log Lik.' 549.76432 (df=4) → -2LL for model
[1] 557.76432 → AIC
[1] 568.06316 → BIC

Random effects:
 Groups Name        Variance Std.Dev.
 ID     (Intercept) 4.3795   2.0927  → extra variance in logits
Number of obs: 97, groups:  ID, 97

Fixed effects:
             Estimate Std. Error z value             Pr(>|z|)
(Intercept)  4.741983   0.433979 10.9268 < 0.00000000000000022 Beta0 logit → prob = .991
NLIvSLI     -1.793161   0.508270 -3.5280            0.0004188 Beta1 still significant
momed12      0.032825   0.134020  0.2449            0.8065140 Beta2 no longer significant

print("Pearson Chi-Square / DF Index of Fit")
sum(residuals(ModelBinAdd, type="pearson")^2)/(N-3)
[1] 0.15402372 → much smaller because all extra variance went into random intercept
```

```
print("Multiv Wald test of model")
BinAddTestR2 = glht(model=ModelBinAdd, linfct=c("NLIvSLI=0","momed12=0"))
print(summary(BinAddTestR2, test=Chisqtest()), digits="8") # Joint chi-square test
Global Test:
      Chisq DF    Pr(>Chisq)
1 14.061888  2 0.00088409677 → much smaller test statistic than before

print("Likelihood Ratio Test for Addition of Random Intercept Variance")
DevTestA=-2*(logLik(ModelBin)-logLik(ModelBinAdd))
RegPvalueA=pchisq((DevTestA), df=1, lower.tail=FALSE); MixPvalueA=RegPvalueA/2
print("Test Statistic, Regular and Mixture P-values for DF=1")
DevTestA;                        RegPvalueA;                          MixPvalueA
'log Lik.' 981.96886 (df=4)    'log Lik.' 1.4914969e-215 (df=4)    'log Lik.' 7.4574845e-216 (df=4)

print("Get odds ratios with 95% CIs")
cbind(OR=exp(ModelBinAdd@beta), exp(confint(ModelBinAdd, parm="beta_", method="Wald")))
                    OR        2.5 %       97.5 %
(Intercept) 114.6613778 48.979525220 268.42300959 exp(Beta0) → was 21.58
NLIvSLI       0.1664332  0.061461097   0.45069174 exp(Beta1) → was 0.29
momed12       1.0333697  0.794653011   1.34379771 exp(Beta2) → was 1.12
```

---

4) Two-Predictor Model with Multiplicative Overdispersion via the Beta-Binomial Distribution

$\#Correct_i \sim BetaBinomial(p_i, Ntrials_i, \phi)$ → p_i is still probability of any one trial being correct

$p_i \sim Beta(a_i, b_i)$ → $a_i = p_i/\phi,\ b_i = (1-p_i)/\phi$

$Logit(p_i \text{ for correct trial}) = \beta_0 + \beta_1(NLIvSLI_i) + \beta_2(MotherEd_i - 12)$

Conditional mean: $\#Correct_i = Ntrials_i * p_i$

Conditional variance of $\#Correct_i = (Ntrials_i * p_i)(1-p_i)[1 + (Ntrials_i - 1)/(\phi + 1)]$

Disclaimer: I struggled to translate this model across the different parameterizations I found, and this formula for the conditional variance produced results that were close to those software provided but not exactly the same…

STATA Syntax and Partial Output for Multiplicative Overdispersion:

```
display "STATA Two-Predictor Beta-Binomial Model with Multiplicative Overdispersion"
display "Using betabin instead that has beta-binomial distribution"
betabin ncorrect c.nlivsli c.momed12, link(logit) n(ntrials) nolog
// LRT for multiplicative overdispersion is done for you automatically
// Pearson chi-square/DF fit not given

Beta-binomial regression                        Number of obs   =         97
Link           = logit                          LR chi2(2)      =      13.61 → LRT for model
Dispersion     = beta-binomial                  Prob > chi2     =     0.0035
Log likelihood = -267.05167                     Pseudo R2       =     0.0248
------------------------------------------------------------------------------
    ncorrect |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     nlivsli |  -.9737565   .2728606    -3.57   0.000    -1.508553   -.4389595 Beta1 still signif
     momed12 |   .0464046   .0685461     0.68   0.498    -.0879434    .1807525 Beta2 now not signif
       _cons |   2.957862   .2500499    11.83   0.000     2.467773     3.44795 Beta0 logit → prob = .951
-------------+----------------------------------------------------------------
    /lnsigma |  -1.421521   .2207495    -6.44   0.000    -1.854182   -.9888596 = log(1/phi)
-------------+----------------------------------------------------------------
       sigma |   .2413467   .0532772                      .156581    .3720007 = 1/phi multiplier in SAS
------------------------------------------------------------------------------
Likelihood-ratio test of sigma=0:  chibar2(01) =  997.63 Prob>=chibar2 = 0.000 → LRT for overdispersion

display "-2LL= " e(ll)*-2  // Print -2LL for model
-2LL= 534.10333

estat ic, n(97)            // Print AIC and BIC
-----------------------------------------------------------------------------
       Model |        Obs  ll(null)  ll(model)      df         AIC        BIC
-------------+---------------------------------------------------------------
           . |         97 -273.8551  -267.0517       4    542.1033   552.4022
-----------------------------------------------------------------------------
```

```
test (c.nlivsli=0)(c.momed12=0)  // Multiv Wald test of model
           chi2(  2) =  14.04
         Prob > chi2 =    0.0009

predict yhatBBpred  // Save predicted propcorrect per real person to dataset
corr yhatBBpred propcorrect  // Get corr of pred and actual propcorrect
display "R2= " r(rho)^2      // Print R2 relative to empty model
R2= .15730629

display "STATA Two-Predictor Binomial Model with Multiplicative Overdispersion"
display "Using betabin and Getting Odds Ratios using eform"
betabin ncorrect c.nlivsli c.momed12, link(logit) n(ntrials) eform nolog

------------------------------------------------------------------------------
    ncorrect |      exp(b)   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     nlivsli |   .3776617    .103049    -3.57   0.000     .2212298    .6447069 exp(Beta1)
     momed12 |   1.047498   .0718019     0.68   0.498     .9158127    1.198119 exp(Beta2)
       _cons |   19.25675   4.815148    11.83   0.000     11.79615     31.4359 exp(Beta0)
-------------+----------------------------------------------------------------
```

R Syntax and Partial Output for Multiplicative Overdispersion:

```
print("R Two-Predictor Binomial Model using vglm with Multiplicative Overdispersion")
ModelBetaBin = vglm(data=Example4a, betabinomial(lmu="logitlink", lrho="logitlink"),
                    formula=cbind(Ncorrect,Nincorrect)~1+NLIvSLI+momed12)
print("Print -2LL, AIC, BIC, and results")
-2*logLik(ModelBetaBin); AIC(ModelBetaBin); BIC(ModelBetaBin); summary(ModelBetaBin)
[1] 534.10333 → -2LL for model
[1] 542.10333 → AIC
[1] 552.40218 → BIC

Coefficients:
              Estimate Std. Error z value             Pr(>|z|)
(Intercept):1  2.957854   0.255237 11.5887 < 0.00000000000000022 Beta0 logit → prob = .951
(Intercept):2 -1.421511   0.214887 -6.6152      0.00000000003711 Log(1/phi) as given in Stata
NLIvSLI       -0.973743   0.271864 -3.5817            0.0003413 Beta1
momed12        0.046404   0.071489  0.6491            0.5162707 Beta2

print("Pearson Chi-Square / DF Index of Fit")
sum(residuals(ModelBetaBin, type="pearson")^2)/(N-3)
[1] 1.7995172 → much closer to 1=good than regular binomial!

print("Multiv Wald test of model")
BinMultTestR2 = glht(model=ModelBetaBin, linfct=c("NLIvSLI=0","momed12=0"))
print(summary(BinMultTestR2, test=Chisqtest()), digits="8") # Joint chi-square test
Global Test:
     Chisq DF   Pr(>Chisq)
1 14.86719  2 0.0005910587

print("Likelihood Ratio Test for Addition of Multiplicative Overdispersion")
DevTestB=-2*(logLik(ModelBin)-logLik(ModelBetaBin))
RegPvalueB=pchisq((DevTestB), df=1, lower.tail=FALSE); MixPvalueB=RegPvalueB/2
print("Test Statistic, Regular and Mixture P-values for DF=1")
DevTestB;        RegPvalueB;         MixPvalueB
[1] 997.62984    [1] 5.8810413e-219  [1] 2.9405206e-219

print("Save predicted propcorrect and correlate with actual propcorrect")
Example4a$PredBetaBin = predict(ModelBetaBin, type="response")
rPredBetaBin = cor.test(Example4a$PredBetaBin, Example4a$PropCorrect, method="pearson")
print("R2"); rPredBetaBin$estimate^2
0.15730603

print("Get odds ratios with 95% CIs")
exp(cbind(OR=coefficients(ModelBin), confint.default(ModelBin)))
                    OR       2.5 %      97.5 %
(Intercept):1 19.25660777 11.67674520 31.75687544 exp(Beta0)
(Intercept):2  0.24134899  0.15839210  0.36775403 1/phi as given in Stata
NLIvSLI        0.37766663  0.22166509  0.64345761 exp(Beta1)
momed12        1.04749703  0.91054552  1.20504687 exp(Beta2)
```

5) Two-Predictor Binomial Model with Zero-Inflation (predicting number incorrect now)

Our negatively skewed data have one-inflation, not zero-inflation, but all the software routines I found were designed only for zero-inflation. So I solved this problem by **predicting number incorrect** instead of number correct. The model below says that number incorrect comes from a binomial distribution that has extra zero values. The "inflation" model that predicts the logit of being an "extra zero" is empty for now, because I just want to see how many extra zeros there are.

$Logit(p_{ip}$ for incorrect trial$) = \beta_{0p} + \beta_{1p}(NLIvsSLI_i) + \beta_{2p}(MotherEd_i - 12)$

$Logit(p_{iz}$ for $y_i = extra\ 0) = \beta_{0z}$

Conditional mean: $\#Incorrect_i = (Ntrials_i * p_{ip}) * p_{iz}$

I'm not even going to try to get the distributional notation or conditional variance right…

STATA Syntax and Partial Output for Binomial with Zero-Inflation (predicting number incorrect):

```
display "STATA Two-Predictor Zero-Inflated Binomial Model"
display "Use zbin and predict nincorrect instead"
display "ilink is for submodel predicting extra 0 (empty here)"
zib nincorrect c.nlivsli c.momed12, link(logit) n(ntrials) ilink(logit) inflate(_cons) nolog
// Pearson chi-square/DF fit not given

Zero-inflated binomial regression               Number of obs   =         97
Regression link: logit                          Nonzero obs     =         57
Inflation link : logit                          Zero obs        =         40
                                                LR chi2(2)      =     126.58 → LRT for model
Log likelihood = -494.1091                      Prob > chi2     =     0.0000
------------------------------------------------------------------------------
  nincorrect |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
nincorrect   |
     nlivsli |   .6787023   .0934716     7.26   0.000     .4955014    .8619033  Beta1p
     momed12 |  -.1148639    .024894    -4.61   0.000    -.1636552   -.0660727  Beta2p
       _cons |  -2.209937   .0825224   -26.78   0.000    -2.371678   -2.048196  Beta0p
-------------+----------------------------------------------------------------
inflate      |
       _cons |  -.3547476   .2063317    -1.72   0.086    -.7591502     .049655  Beta0z logit of extra 0
------------------------------------------------------------------------------  → probability = .4122

display "-2LL= " e(ll)*-2  // Print -2LL for model
-2LL= 988.2183

estat ic, n(97)            // Print AIC and BIC
-----------------------------------------------------------------------------
       Model |        Obs  ll(null)  ll(model)      df         AIC        BIC
-------------+---------------------------------------------------------------
           . |         97 -557.3991  -494.1091       4    996.2183   1006.517
-----------------------------------------------------------------------------
test (c.nlivsli=0)(c.momed12=0)  // Multiv Wald test of model
           chi2(  2) =  116.04
         Prob > chi2 =    0.0000

display "LRT for zero-inflation (binomial vs zero-inflated binomial)"
display "Test Statistic (df=1)= "          1531.7332-988.2183
Test Statistic (df=1)= 543.5149
display "Regular p-value= "     (1-chi2(1, 1531.7332-988.2183))
0
display "Mixture p-value= " 0.5*(1-chi2(1, 1531.7332-988.2183))
0

display "STATA Two-Predictor Zero-Inflated Binomial Model"
display "Use zbin and predict nincorrect instead, get odds ratios"
display "ilink is for submodel predicting extra 0 (empty here)"
zib nincorrect c.nlivsli c.momed12, link(logit) n(ntrials) ilink(logit) inflate(_cons) eform nolog
------------------------------------------------------------------------------
  nincorrect |     exp(b)   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
nincorrect   |
     nlivsli |   1.971318   .1842622     7.26   0.000     1.641321    2.367663 exp(Beta1p)
     momed12 |   .8914875   .0221927    -4.61   0.000     .8490347    .9360629 exp(Beta2p)
       _cons |   .1097075   .0090533   -26.78   0.000      .093324    .1289673 exp(Beta0p)
```

```
-------------+----------------------------------------------------------------
inflate      |
       _cons |  -.3547476   .2063317    -1.72   0.086    -.7591502     .049655  Beta0z logit of extra 0
------------------------------------------------------------------------------  → probability = .4122
```

R Syntax and Partial Output for Binomial with Zero-Inflation (predicting number incorrect):

```
print("R Two-Predictor Zero-Inflated Binomial Model using vglm Predicting Nincorrect")
ModelZBin = vglm(data=Example4a, zibinomialff(lprob="logitlink", lonempstr0="logitlink",
                 multiple.responses=FALSE, ionempstr0=NULL, zero="onempstr0"),
                 formula=cbind(Nincorrect,Ncorrect)~1+NLIvSLI+momed12)
print("Print -2LL, AIC, BIC, and results")
-2*logLik(ModelZBin); AIC(ModelZBin); BIC(ModelZBin); summary(ModelZBin)
[1] 988.2183 → -2LL for model
[1] 996.2183 → AIC
[1] 1006.5171 → BIC

Coefficients:
               Estimate Std. Error  z value             Pr(>|z|)
(Intercept):1 -2.209921   0.068295 -32.3583 < 0.00000000000000022  Beta0p
(Intercept):2  0.354748   0.206329   1.7193              0.08555  Beta0z*-1 = logit of not extra 0
NLIvSLI        0.678734   0.084490   8.0333 0.0000000000000009488  Beta1p
momed12       -0.114872   0.022742  -5.0512 0.0000004391152171797  Beta2p

Names of linear predictors: logitlink(prob), logitlink(onempstr0)
Log-likelihood: -494.10915 on 190 degrees of freedom

print("Pearson Chi-Square / DF Index of Fit")
sum(residuals(ModelZBin, type="pearson")^2)/(N-3)
[1] 13.359286 → way too high again!

print("Multiv Wald test of model")
ZBinMultTestR2 = glht(model=ModelZBin, linfct=c("NLIvSLI=0","momed12=0"))
print(summary(ZBinMultTestR2, test=Chisqtest()), digits="8") # Joint chi-square test
Global Test:
     Chisq DF   Pr(>Chisq)
1 111.55444  2 5.9739459e-25 → way too optimistic given bad distribution fit

print("Likelihood Ratio Test for Addition of Zero-Inflation over Binomial")
DevTestC=-2*(logLik(ModelBin)-logLik(ModelZBin))
RegPvalueC=pchisq((DevTestC), df=1, lower.tail=FALSE); MixPvalueC=RegPvalueC/2
print("Test Statistic, Regular and Mixture P-values for DF=1")
DevTestC;     RegPvalueC;          MixPvalueC
1] 543.51488  [1] 3.2417789e-120   [1] 1.6208894e-120

print("Get odds ratios with 95% CIs")
exp(cbind(OR=coefficients(ModelZBin), confint.default(ModelZBin)))

                     OR       2.5 %     97.5 %
(Intercept):1 0.10970928 0.095964415 0.12542281 exp(Beta0)
(Intercept):2 1.42582081 0.951561912 2.13645058 exp(Beta0z*-1)
NLIvSLI       1.97138047 1.670522825 2.32642193 exp(Beta1)
momed12       0.89148032 0.852617086 0.93211498 exp(Beta2)
```

---

6) Two-Predictor Beta-Binomial Model with Zero-Inflation (predicting number incorrect)

The model below says that number incorrect comes from a beta-binomial distribution that has extra zero values (instead of a binomial distribution that has extra zero values), allowing multiplicative overdispersion.

$Logit(p_{ip} \text{ for incorrect}) = \beta_{0p} + \beta_{1p}(NLIvSLI_i) + \beta_{2p}(MotherEd_i - 12)$

$Logit(p_{iz} \text{ for } y_i = extra\ 0) = \beta_{0z}$

Conditional mean: $\#Incorrect_i = (Ntrials_i * p_{ip}) * p_{iz}$

I'm not even going to try to get the distributional notation or conditional variance right…

```
# I could not find zero-inflated beta-binomial regression in R, so I gave up
```

STATA Syntax and Partial Output for Beta-Binomial with Zero-Inflation (empty inflation model):

```
display "STATA Two-Predictor Zero-Inflated Beta-Binomial Model"
display "Use zibbin and predict nincorrect instead"
zibbin nincorrect c.nlivsli c.momed12, link(logit) n(ntrials) ilink(logit) inflate(_cons) nolog
// Pearson chi-square/DF fit not given

Zero-inflated beta-binomial regression          Number of obs   =         97
Regression link: logit                          Nonzero obs     =         57
Inflation link : logit                          Zero obs        =         40
                                                LR chi2(2)      =      11.61 → LRT of model
Log likelihood =  -263.789                      Prob > chi2     =     0.0030
------------------------------------------------------------------------------
  nincorrect |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
nincorrect   |
    nlivssli |   1.128224   .3464563     3.26   0.001     .4491825    1.807266 Beta1p
     momed12 |  -.0178967   .0894132    -0.20   0.841    -.1931434    .1573499 Beta2p
       _cons |  -2.750534   .3270209    -8.41   0.000    -3.391483   -2.109585 Beta0p logit → prob = .06
-------------+----------------------------------------------------------------
inflate      |                                                                 Beta0z logit of extra 0
       _cons |  -1.095397   .4369649    -2.51   0.012    -1.951832   -.2389614 → probability = .2506
-------------+----------------------------------------------------------------
    /lnsigma |  -1.870879   .2495082                     -2.359906   -1.381852 → log(1/phi)
-------------+----------------------------------------------------------------
       sigma |   .1539883   .0384213                      .0944291    .2511131 → 1/phi multiplier in SAS
------------------------------------------------------------------------------
display "-2LL= " e(ll)*-2  // Print -2LL for model
-2LL= 527.57802

estat ic, n(97)            // Print AIC and BIC
-----------------------------------------------------------------------------
       Model |        Obs  ll(null)  ll(model)      df         AIC        BIC
-------------+---------------------------------------------------------------
           . |         97 -269.5932   -263.789       5     537.578   550.4516
-----------------------------------------------------------------------------
test (c.nlivsli=0)(c.momed12=0)  // Multiv Wald test of model
           chi2(  2) =  12.43
         Prob > chi2 =   0.0020

display "LRT for overdispersion (zero-inflated: binomial vs beta-binomial)"
display "Test Statistic (df=1)= "         988.2183-527.5780
Test Statistic (df=1)= 460.6403 → definitely need multiplicative overdispersion
display "Regular p-value= "     (1-chi2(1,988.2183-527.5780))
0
display "Mixture p-value= " 0.5*(1-chi2(1,988.2183-527.5780))
0

// Save predicted propincorrect per real person to dataset
predict yhatZIBB, xb              // Predicted outcome in logits
gen Npred=1/(1+exp(-1*yhatZIBB)) // Convert to probability
corr Npred propincorrect          // Get corr of pred and actual propincorrect
display "R2=" r(rho)^2            // Print R2 relative to empty model
R2= .14449233

display "STATA Two-Predictor Zero-Inflated Beta-Binomial Model"
display "Use zibbin and predict nincorrect instead, get odds ratios"
zibbin nincorrect c.nlivsli c.momed12, link(logit) n(ntrials) ilink(logit) inflate(_cons) eform nolog

------------------------------------------------------------------------------
  nincorrect |     exp(b)   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
nincorrect   |
     nlivsli |   3.090165   1.070607     3.26   0.001     1.567031    6.093766 exp(Beta1p)
     momed12 |   .9822625   .0878272    -0.20   0.841     .8243638    1.170405 exp(Beta2p)
       _cons |   .0638938   .0208946    -8.41   0.000     .0336587    .1212883 exp(Beta0p)
-------------+----------------------------------------------------------------
inflate      |                                                                 Beta0z logit of extra 0
       _cons |  -1.095397   .4369649    -2.51   0.012    -1.951832   -.2389614 → probability = .2506
-------------+----------------------------------------------------------------
    /lnsigma |  -1.870879   .2495082                     -2.359906   -1.381852 → log(1/phi)
```

```
-------------+----------------------------------------------------------------
       sigma |   .1539883   .0384213                      .0944291    .2511131 → 1/phi multiplier in SAS
```

---

7) Four-Predictor Beta-Binomial Model with Zero-Inflation (now with predictors in inflation model)

This model adds our two predictors to the zero-inflation model (customizing the probability of being an extra zero).

$Logit(p_i \text{ for incorrect}) = \beta_{0p} + \beta_{1p}(NLIvsSLI_i) + \beta_{2p}(MotherEd_i - 12)$
$Logit(p_{iz} \text{ for } y_i = extra\ 0) = \beta_{0z} + \beta_{1z}(NLIvsSLI_i) + \beta_{2z}(MotherEd_i - 12)$
Conditional mean: $\#Incorrect_i = (Ntrials_i * p_i) * p_{iz}$

> I'm not even going to try to get the distributional notation or conditional variance right…

STATA Syntax and Partial Output for Beta-Binomial with Zero-Inflation (predictor inflation model):

```
display "STATA Two-Predictor Zero-Inflated Beta-Binomial Model"
display "Switch to zibbin and predict nincorrect instead"
display "Add two predictors of being extra zero"
zibbin nincorrect c.nlivsli c.momed12, link(logit) n(ntrials) ///
                  ilink(logit) inflate(c.nlivsli c.momed12) nolog

Zero-inflated beta-binomial regression            Number of obs   =         97
Regression link: logit                            Nonzero obs     =         57
Inflation link : logit                            Zero obs        =         40
                                                  LR chi2(2)      =       7.38
Log likelihood = -261.8274                        Prob > chi2     =     0.0249
------------------------------------------------------------------------------
  nincorrect |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
nincorrect   |
    nlivssli |   .3036772   .3546852     0.86   0.392     -.391493    .9988474  Beta1p
     momed12 |  -.2189386   .0812336    -2.70   0.007    -.3781535   -.0597237  Beta2p
       _cons |  -2.173967   .3963158    -5.49   0.000    -2.950731   -1.397202  Beta0p
-------------+----------------------------------------------------------------
inflate      |                                                                 → predict logit of extra 0
    nlivssli |  -3.970179   5.512301    -0.72   0.471    -14.77409    6.833733  Beta2z
     momed12 |  -.9569979   1.428802    -0.67   0.503    -3.757398    1.843402  Beta1z
       _cons |   .0198758   .6209887     0.03   0.974     -1.19724    1.236991  Beta0z
-------------+----------------------------------------------------------------
    /lnsigma |  -1.652934   .3139631                     -2.26829   -1.037578 → log(1/phi)
-------------+----------------------------------------------------------------
       sigma |   .1914873   .0601199                      .103489     .354312 → 1/phi multiplier in SAS
------------------------------------------------------------------------------
display "-2LL= " e(ll)*-2  // Print -2LL for model
-2LL= 523.65476

display "STATA Two-Predictor Zero-Inflated Beta-Binomial Model"
display "Switch to zibbin and predict nincorrect instead, get odds ratios"
display "Add two predictors of being extra zero"
zibbin nincorrect c.nlivsli c.momed12, link(logit) n(ntrials) ///
                  ilink(logit) inflate(c.nlivsli c.momed12) eform nolog
------------------------------------------------------------------------------
  nincorrect |     exp(b)   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
nincorrect   |
     nlivsli |   1.354832   .4805387     0.86   0.392     .6760468    2.715151 exp(Beta1p)
     momed12 |   .8033711   .0652607    -2.70   0.007     .6851254    .9420248 exp(Beta2p)
       _cons |   .1137256   .0450713    -5.49   0.000     .0523014    .2472879 exp(Beta0p)
-------------+----------------------------------------------------------------
inflate      |
     nlivsli |  -3.970179   5.512301    -0.72   0.471    -14.77409    6.833733 exp(Beta1z)
     momed12 |  -.9569979   1.428802    -0.67   0.503    -3.757398    1.843402 exp(Beta2z)
       _cons |   .0198758   .6209887     0.03   0.974     -1.19724    1.236991 exp(Beta0z)
-------------+----------------------------------------------------------------
    /lnsigma |  -1.652934   .3139631                     -2.26829   -1.037578 → log(1/phi)
-------------+----------------------------------------------------------------
       sigma |   .1914873   .0601199                      .103489     .354312 → 1/phi multiplier in SAS
------------------------------------------------------------------------------
```

So which one should we pick? Let's do some informal model comparisons using distribution fit and relative fit (* may not be exactly comparable due to differences in estimation technique, but they should be close)

Two-Predictor Model	Pearson Chi-Square / DF	−2LL*	AIC*	BIC*
2. Regular Binomial	15.41	1531.73	1537.73	1545.46
3. +Additive Overdispersion	0.15	549.76	557.76	568.10
4. Beta-Binomial (multiplicative)	1.80	534.10	542.10	552.40
5. Zero-Inflated Binomial	13.36	988.22	996.22	1006.52
6. Zero-Inflated Beta-Binomial	**(SAS) 0.87**	**527.58**	**537.58**	**550.45**
7. ZIBB + Predictors	?	523.65	537.65	555.68

Sample results using all programs (final model = zero-inflated beta-binomial without inflation predictors):

The extent that grammatical understanding (measured either as percent correct or percent incorrect; see below) at third grade could be predicted by language impairment group (non-specific=0, specific=1) and mother's years of education (centered such that 0=12 years) was examined in a series of generalized linear models. In the sample of $N = 97$ children, the mean percent correct was 0.92, with a large percentage of observations at or near the ceiling of 1. Accordingly, we predicted the number of correct trials out of the number of possible trials using a logit link function to keep the predicted proportion correct outcome bounded at 1. The type of model specifies that the number of correct responses follows a binomial-based distribution with 100 total trials and the model predicts the logit (log-odds) of a correct answer for any trial. Predicted outcomes in a logit metric can be translated into proportion correct via an inverse link function (which provides model-predicted proportions and their standard errors). All models were estimated using maximum likelihood within SAS GLIMMIX and FMM to assess distribution fit (or Stata glm, betabin, zib, and zibbin, or R vglm and glmer); predictor fixed effects were tested univariately using z-distributions without denominator degrees of freedom. Effect sizes are provided below as odds ratios: the exponentiated logit coefficient in which values from 0 to 1 indicate negative associations, 1 indicates no association, and values above 1 indicate positive associations.

Before interpreting our results, we tested the fit of models with alternative binomial-based conditional outcome distributions (each with main effects of group and mother's education) by examining the Pearson χ^2/DF statistic (in which 1=good fit), as well as likelihood ratio tests (i.e., treating −2 times the difference in log-likelihood between nested models as a χ^2 statistic with degrees of freedom equal to the number of additional parameters). As expected given the negatively skewed observed distribution, a model specifying a standard binomial distribution for number correct did not fit well (Pearson $\chi^2/DF = 15.41$). Two methods of allowing overdispersion were then examined. First, we allowed additive overdispersion via an observation-level random intercept, which significantly improved model fit, $-2\Delta LL(1) = 987.97$, $p < .0001$, but created a tendency towards underdispersion (Pearson $\chi^2/DF = 0.15$). Second, we allowed multiplicative overdispersion by using a beta-binomial distribution, which significantly improved model fit, $-2\Delta LL(1) = 997.63$, $p < .0001$, and appeared to fit well (Pearson $\chi^2/DF = 1.80$). We then examined the potential for one-inflation by predicting number *incorrect* instead so that zero-inflation models could be used. A model predicting number incorrect with a zero-inflated binomial distribution was examined but did not fit as well (Pearson $\chi^2/DF = 13.36$), although using a zero-inflated beta-binomial distribution instead did result in good fit (Pearson $\chi^2/DF = 0.87$), as well as the lowest AIC and BIC of all the models. We also examined group and mother's education as predictors of zero-inflation but neither was significant (with higher AIC and BIC values), and thus the empty (unconditional) zero-inflation model was retained.

The model results indicated that 25.06% of the sample were predicted to be an extra 0 (i.e., to be part of the zero-inflated component of the distribution for number incorrect). Otherwise, the predicted intercept for a child with non-specific language impairment whose mother had 12 years of education was a logit = −2.75, which translates into percent incorrect = 0.06. Children with specific language impairment were predicted to have significantly more incorrect responses (logit = 1.12, OR = 3.09), although no significant difference was found for mother's years of education (logit = −0.02, OR = 0.98). The scale parameter for multiplicative overdispersion was 1/0.15, which was significant, $-2\Delta LL(1) = 460.64$, $p < .0001$, relative to the zero-inflated binomial alternative.